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On a moment problem for rational matrix-valued functions

Bernd Fritzsche*, Bernd Kirstein, Andreas Lasarow

Fakultät für Mathematik und Informatik, Mathematisches Institut, Universität Leipzig, Augustusplatz 10, D-04109 Leipzig, Germany

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Abstract

The papers deals with a finite moment problem for rational matrix-valued functions. We present a necessary and sufficient condition for the solvability of the problem. In the nondegenerate case we construct a particular solution which has interesting extremal properties.

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0. Introduction

The main goal of this paper is to study a moment problem for matrix-valued rational functions, which generalizes the truncated trigonometric matrix moment problem. From another point of view the matrix moment problem in question generalizes some investigations done by Bultheel et al. [3] in the context of the theory of orthogonal rational functions to the matrix case. In [9] the authors have gone first steps towards constructing a matrix generalization of the theory of orthogonal rationals functions created by Bultheel et al. [3]. There were obtained several results on rank invariance of various Gramian matrices. A closer look at

* Corresponding author. Tel.: +49-341-9732106; fax: +49-341-9732198.
E-mail address: fritzsche@mathematik.uni-leipzig.de (B. Fritzsche).

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these Gramian matrices led us to the rational matrix moment problem studied in this paper.

A crucial part in the generalization from the scalar to the matrix case is the definition of the spaces for which an orthogonal basis is to be constructed. This topic is handled in Section 1. Starting from a sequence of complex numbers which are not located at the unit circle $\mathbb{T}$ of the complex plane we form corresponding modules of rational matrix-valued functions with prescribed poles. With the aid of a nonnegative Hermitian Borel measure on $\mathbb{T}$ we introduce simultaneously left and right matrix-valued inner product structures on these modules. In this way, we are led to a natural inverse problem. Starting from such a module of rational matrix-valued functions and a given block matrix $\mathbf{G}$ the problem is to determine the set of all nonnegative Hermitian Borel measures F on $\mathbb{T}$ having $\mathbf{G}$ as Gramian matrix. The exact formulation of this matrix moment problem will be given in Section 2. The main result of Section 2 is a necessary and sufficient condition for its solvability (see Proposition 5). This criterion is essentially based on reduction to an appropriately built trigonometric matrix moment problem.

Similarly as in [3] by Bultheel et al. for the scalar case or in [4] by Delsarte et al. for the case of matrix polynomials, starting with Section 3 we will mainly concentrate on those matrix measures which produce nondegenerate left and right matrix-valued inner product structures in the modules in question, i.e., we will study moment problems with a given positive Hermitian moment matrix. Our corresponding investigations are guided in some aspects by the work of Delsarte et al. on the matrix polynomial case (see [4,5]). Studying extremal problems for particularly constructed functionals for matrix polynomials these authors were led to distinguished matrix polynomials which served as starting point for the construction of a particular nonnegative Hermitian Borel measure on $\mathbb{T}$. This matrix measure turns out to be a solution of the truncated trigonometric matrix moment problem under consideration which has moreover some extremal properties amongst the set of all solutions. Extending the techniques of Delsarte et al. to the case of rational matrix-valued functions we obtain a full rational generalization of their result.

By the way, in Section 4 we study several extremal problems for rational matrix-valued functions. These problems turn out to have unique solutions. The minimizing rational matrix-valued functions serve not only as starting point for the construction of the announced particular solution of our rational moment problem. Moreover, in subsequent work we will indicate that these rational functions are an essential tool in our construction of a theory of orthogonal rational matrix-valued functions.

In addition to the strategy of Delsarte et al., our methods make extensive use of reproducing kernels in Hilbert modules. An essential step in constructing the distinguished solution of our moment problem is Theorem 25 which contains a key observation about the location of zeros of the reproducing kernels.

1. Notation and basic facts on rational matrix-valued functions

Throughout this paper, let p, q and r belong to the set $\mathbb{N}$ of all positive integers. Let us use $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ and $\mathbb{N}_0$ to denote the sets of all complex numbers, of all real numbers, of all integers, and of all nonnegative integers, respectively. If $m \in \mathbb{N}_0$ and if $n \in \mathbb{N}_0$ or $n = +\infty$, then we will write $\mathbb{N}_{m,n}$ for the set of all integers k which satisfy $m \leqslant k \leqslant n$. If $\mathfrak{X}$ is a nonempty set, then let $\mathfrak{X}^{p\times q}$ be the set of $p \times q$ matrices each entry of which belongs to $\mathfrak{X}$. The notation $\mathbf{0}_{p\times q}$ stands for the null matrix that belongs to the set $\mathbb{C}^{p\times q}$ of all complex $p \times q$ matrices, and the identity matrix which belongs to $\mathbb{C}^{q\times q}$ will be denoted by $\mathbf{I}_q$. If the size of the null matrix or the identity matrix is clear, we will omit the indexes. If $\mathbf{A} \in \mathbb{C}^{q\times q}$, then $\operatorname{tr}\mathbf{A}$ indicates the trace of $\mathbf{A}$. Further, for all $\mathbf{A} \in \mathbb{C}^{q\times q}$, let $\operatorname{Re}\mathbf{A}$ and $\operatorname{Im}\mathbf{A}$ designate the real part of $\mathbf{A}$ and the imaginary part of $\mathbf{A}$, respectively: $\operatorname{Re}\mathbf{A} := \frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$ and $\operatorname{Im}\mathbf{A} := \frac{1}{2\mathrm{i}}(\mathbf{A} - \mathbf{A}^*)$.

The set of all nonnegative Hermitian complex $q \times q$ matrices will be designated by $\mathbb{C}_{\geqslant}^{q\times q}$. If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian complex $q \times q$ matrices, then $\mathbf{A} \geqslant \mathbf{B}$ (or $\mathbf{B} \leqslant \mathbf{A}$) means $\mathbf{A} - \mathbf{B}$ is nonnegative Hermitian. The symbol $\mathbb{T}$ stands for the unit circle, $\mathbb{D}$ for its interior and $\mathbb{E}$ for its exterior with respect to the extended complex plane $\mathbb{C}_0 := \mathbb{C} \cup \{\infty\}$, i.e., $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{E} := \mathbb{C}_0 \backslash (\mathbb{D} \cup \mathbb{T})$. The constant function defined on $\mathbb{C}_0$ (respectively, $\mathbb{C}$) with value $\mathbf{0}_{p\times q}$ is denoted by $O_{p\times q}$. Let $\mathfrak{B}_{p\times q}$ (respectively, $\mathfrak{B}_1$) be the σ-algebra of all Borel subsets of $\mathbb{C}^{p\times q}$ (respectively, $\mathbb{C}$), and let $\mathfrak{B}_{\mathbb{T}} := \mathfrak{B}_1 \cap \mathbb{T}$. If Y is a nonempty subset of a set $\mathfrak{X}$ and if f is a mapping defined on $\mathfrak{X}$, then we will use $\operatorname{Rstr.}_Y f$ to denote the restriction of f onto Y. If f is a matrix-valued function defined on a subset M of $\mathbb{C}_0$ with $\mathbb{T} \subseteq M$, then $\underline{f}$ is short for $\operatorname{Rstr.}_{\mathbb{T}} f$.

Let Λ be a nonempty set, and let $\mathfrak{A}$ be a σ-algebra in Λ. A mapping F whose domain is $\mathfrak{A}$, and whose values belong to $\mathbb{C}_{\geqslant}^{q\times q}$ is called $q \times q$ nonnegative Hermitian measure on $(\Lambda, \mathfrak{A})$ if it is countably additive, i.e., if F satisfies

$$F\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} F(A_k)$$

for every infinite sequence $(A_k)_{k=1}^{\infty}$ of pairwise disjoint sets which belong to $\mathfrak{A}$. Let us use $\mathscr{M}_{\geqslant}^{q}(\Lambda, \mathfrak{A})$ for the set of all $q \times q$ nonnegative Hermitian measures on $(\Lambda, \mathfrak{A})$. Let $F = (F_{jk})_{j,k=1}^{q} \in \mathscr{M}_{\geqslant}^{q}(\Lambda, \mathfrak{A})$. Then F is absolutely continuous with respect to the trace measure $\tau F := \Sigma_{j=1}^{q} F_{jj}$ of F. The matrix-valued function $F'_\tau = \left(\frac{\mathrm{d}F_{jk}}{\mathrm{d}\tau F}\right)_{j,k=1}^{q}$ built from the Radon–Nikodym derivatives $\frac{\mathrm{d}F_{jk}}{\mathrm{d}\tau F}$ of F_{jk} with respect to τF is called the trace derivative of F and satisfies $\operatorname{tr} F'_\tau = 1$ and $\mathbf{0}_{q\times q} \leqslant F'_\tau \leqslant \mathbf{I}_q$ τF-almost everywhere on Λ. If ν is a measure on $(\Lambda, \mathfrak{A})$ and $s \in (0, \infty)$, then $\mathscr{L}^s(\Lambda, \mathfrak{A}, \nu; \mathbb{C})$ stands for the set of all complex-valued $\mathfrak{A} - \mathfrak{B}_1$-measurable functions f which are defined on Λ and for which $|f|^s$ is integrable with respect to the measure ν. An ordered pair $[\Phi, \Psi]$ consisting of an $\mathfrak{A} - \mathfrak{B}_{q\times p}$-measurable matrix-valued function $\Phi : \Lambda \to \mathbb{C}^{q\times p}$ and an $\mathfrak{A} - \mathfrak{B}_{q\times r}$-measurable matrix-valued function $\Psi : \Lambda \to \mathbb{C}^{q\times r}$ is called right-integrable with respect to F if $\Phi^* F'_\tau \Psi$ belongs to $[\mathscr{L}^1(\Lambda, \mathfrak{A}, \tau F; \mathbb{C})]^{p\times r}$. In this case, we set

$$\int_A \Phi^* \, \mathrm{d}F \, \Psi := \int_A \Phi^* F'_\tau \Psi \, \mathrm{d}\tau F$$

for each $A \in \mathfrak{A}$. If ν is a σ-finite measure on $(\Lambda, \mathfrak{A})$ such that F is absolutely continuous with respect to ν, then

$$\int_A \Phi^* \, \mathrm{d}F \, \Psi = \int_A \Phi^* F'_\nu \Psi \, \mathrm{d}\nu$$

for all $A \in \mathfrak{A}$, where $F'_\nu = \left(\frac{\mathrm{d}F_{jk}}{\mathrm{d}\nu}\right)_{j,k=1}^q$ is the Radon–Nikodym derivative of F with respect to ν. An ordered pair $[\Phi, \Psi]$ consisting of an $\mathfrak{A} - \mathfrak{B}_{p\times q}$-measurable matrix-valued function $\Phi : \Lambda \to \mathbb{C}^{p\times q}$ and an $\mathfrak{A} - \mathfrak{B}_{r\times q}$-measurable matrix-valued function $\Psi : \Lambda \to \mathbb{C}^{r\times q}$ is said to be left-integrable with respect to F if $[\Phi^*, \Psi^*]$ is right-integrable with respect to F. The set of all matrix-valued functions $\Phi : \Lambda \to \mathbb{C}^{q\times p}$ for which the pair $[\Phi, \Phi]$ is right-integrable with respect to F is denoted by $q \times p - \mathscr{L}^2_R(\Lambda, \mathfrak{A}, F)$, whereas the set of all matrix-valued functions $\Phi : \Lambda \to \mathbb{C}^{p\times q}$ for which $[\Phi, \Phi]$ is left-integrable with respect to F is designated by $p \times q - \mathscr{L}^2_L(\Lambda, \mathfrak{A}, F)$. Observe that $q \times p - \mathscr{L}^2_R(\Lambda, \mathfrak{A}, F)$ (respectively, $p \times q - \mathscr{L}^2_L(\Lambda, \mathfrak{A}, F)$) is a right (respectively, left) $\mathbb{C}^{p\times p}$-semi-Hilbert module, where the Gramian structure is given by

$$(\Phi, \Psi)_F := \int_\Lambda \Phi^* \, \mathrm{d}F \Psi \quad \left(\text{respectively, } (\Phi, \Psi)_F := \int_\Lambda \Phi \, \mathrm{d}F \Psi^*\right)$$

for all matrix-valued functions Φ and Ψ which belong to $q \times p - \mathscr{L}^2_R(\Lambda, \mathfrak{A}, F)$ (respectively, $p \times q - \mathscr{L}^2_L(\Lambda, \mathfrak{A}, F)$). For more details of the integration theory with respect to nonnegative Hermitian measures we refer to Rosenberg [11–13] (see also [9, Section 1]).

Particularly, we will turn our attention to the set $\mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Any measure $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ is uniquely determined by the sequence $(\mathbf{\Gamma}^{(F)}_j)_{j\in\mathbb{Z}}$ of its Fourier coefficients

$$\mathbf{\Gamma}^{(F)}_j := \int_\mathbb{T} z^{-j} F(\mathrm{d}z), \quad j \in \mathbb{Z}, \tag{1}$$

where

$$\mathbf{\Gamma}^{(F)}_j = \int_\mathbb{T} (z^j \mathbf{I}_q)^* F(\mathrm{d}z) \mathbf{I}_q \tag{2}$$

holds obviously for each $j \in \mathbb{Z}$. The matricial version of a famous theorem due to G. Herglotz says that a given sequence $(\mathbf{\Gamma}_j)_{j\in\mathbb{Z}}$ of complex $q \times q$ matrices is the sequence of Fourier coefficients of some measure $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ if and only if the sequence $(\mathbf{\Gamma}_j)_{j\in\mathbb{Z}}$ is nonnegative definite, i.e., if and only if for each $n \in \mathbb{N}_0$ the block Toeplitz matrix $(\mathbf{\Gamma}_{j-k})^n_{j,k=0}$ is nonnegative Hermitian. In particular, if $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$, then for each $n \in \mathbb{N}_0$ the block Toeplitz matrix

$$\mathbf{T}_n^{(F)} := \left(\boldsymbol{\Gamma}_{j-k}^{(F)}\right)_{j,k=0}^{n} \tag{3}$$

of order $n+1$ associated with F is nonnegative Hermitian.

Now starting from a sequence of complex numbers we define modules of rational matrix-valued functions with prescribed pole structure. In a slight extension of the situation considered by Bultheel et al. in [3] the rational functions we consider are allowed to have poles not only in $\mathbb{C}_0 \setminus (\mathbb{D} \cup \mathbb{T})$, but also in the open unit disk $\mathbb{D}$.

Let $m \in \mathbb{N}$ or let $m = +\infty$, and let $(\alpha_j)_{j=1}^m$ be a fixed sequence of complex numbers. Further, let $\pi_{\alpha,0} : \mathbb{C}_0 \to \mathbb{C}$ be the constant function with value 1, and let $\mathscr{R}_{\alpha,0}$ denote the set of all constant complex-valued functions defined on $\mathbb{C}_0$. For each positive integer n with $n \leqslant m$, let $\pi_{\alpha,n} : \mathbb{C} \to \mathbb{C}$ be defined by

$$\pi_{\alpha,n}(w) := \prod_{j=1}^{n} (1 - \overline{\alpha_j} w) \tag{4}$$

and let $\mathscr{R}_{\alpha,n}$ designate the set of all complex-valued functions f which are rational and which admit a representation $f = \frac{1}{\pi_{\alpha,n}} P$ with some complex polynomial P of degree not greater than n. Every function which belongs to $\mathscr{R}_{\alpha,n}$ is holomorphic in $\mathbb{C}_0 \setminus P_{\alpha,n}$ where $P_{\alpha,n} := \bigcup_{j=1}^{n} \left\{ \frac{1}{\overline{\alpha_j}} \right\}$, $\bigcup_{j=1}^{0} \left\{ \frac{1}{\overline{\alpha_j}} \right\} := \emptyset$, and $\frac{1}{0} := \infty$. Obviously, if $n < m$, then $\mathscr{R}_{\alpha,n}$ does not depend on the numbers α_j, $j \in \mathbb{N}_{n+1,m}$. Identifying constant complex-valued functions defined on $\mathbb{C}_0$ with their restrictions onto $\mathbb{C}$, one can easily see that in the case that $\alpha_j = 0$ holds for all $j \in \mathbb{N}_{1,n}$ the class $\mathscr{R}_{\alpha,n}$ coincides with the set $\mathscr{P}_n$ of all complex-valued polynomials of degree not greater than n. If n_1 and n_2 are integers with $0 \leqslant n_1 \leqslant n_2 \leqslant m$, then $\mathscr{R}_{\alpha,n_1} \subseteq \mathscr{R}_{\alpha,n_2}$. If a sequence $(\alpha_j)_{j \in \mathbb{N}}$ of complex numbers is given, then let $\mathscr{R}_\alpha := \bigcup_{n \in \mathbb{N}_0} \mathscr{R}_{\alpha,n}$. Each function $f \in \mathscr{R}_\alpha^{p \times q}$ is holomorphic in $\mathbb{C}_0 \setminus P_\alpha$ where $P_\alpha := \bigcup_{j=1}^{\infty} \left\{ \frac{1}{\overline{\alpha_j}} \right\}$. The classes $\mathscr{R}_{\alpha,n}^{p \times q}$ and $\mathscr{R}_\alpha^{p \times q}$ are considered in [9, Section 2]. For the convenience of the reader, we recall some essential facts which will be used below. If $n \in \mathbb{N}_0$, then let us first observe that the class $\mathscr{R}_{\alpha,n}^{p \times q}$ can be considered as right $\mathbb{C}^{q \times q}$-submodule of the right $\mathbb{C}^{q \times q}$-module $\mathscr{R}_\alpha^{p \times q}$. On the other hand, for each $n \in \mathbb{N}_0$, the class $\mathscr{R}_{\alpha,n}^{p \times q}$ is also a left $\mathbb{C}^{p \times p}$-submodule of the left $\mathbb{C}^{p \times p}$-module $\mathscr{R}_\alpha^{p \times q}$.

Remark 1. Let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence of complex numbers and let $n \in \mathbb{N}_0$. For each $k \in \mathbb{N}_{0,n}$, let $E_{k,q}$ be the $q \times q$ matrix polynomial which is given by

$$E_{k,q}(w) := w^k \mathbf{I}_q \tag{5}$$

for each $w \in \mathbb{C}$. Then the system $\left\{ \frac{1}{\pi_{\alpha,n}} E_{k,q} : k \in \mathbb{N}_{0,n} \right\}$ is both a basis of the right $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$ and a basis of the left $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$.

Assume that F belongs to $\mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C} \setminus \mathbb{T}$, and let $n \in \mathbb{N}_0$. For every choice of $X \in \mathscr{R}_{\alpha,n}^{q \times p}$ and

$Y \in \mathscr{R}_{\alpha,n}^{q\times p}$, the ordered pair $[\underline{X}, \underline{Y}]$ is right-integrable with respect to F (see, e.g., [9, Proposition 3.2]). If

$$(X, Y)_r := \int_{\mathbb{T}} \underline{X}^* \, \mathrm{d}F \underline{Y} \tag{6}$$

for each $X \in \mathscr{R}_{\alpha,n}^{q\times p}$ and each $Y \in \mathscr{R}_{\alpha,n}^{q\times p}$, then it is readily checked that $\left(\mathscr{R}_{\alpha,n}^{q\times p}, (\cdot,\cdot)_r\right)$ is a right $\mathbb{C}^{p\times p}$-semi-Hilbert module. Similarly, one can see that $\left(\mathscr{R}_{\alpha,n}^{p\times q}, (\cdot,\cdot)_l\right)$ is a left $\mathbb{C}^{p\times p}$-semi-Hilbert module where

$$(X, Y)_l := \int_{\mathbb{T}} \underline{X} \, \mathrm{d}F \underline{Y}^* \tag{7}$$

for every choice of X and Y in $\mathscr{R}_{\alpha,n}^{p\times q}$.

Now we turn our attention to the case $p = q$. If $(X_j)_{j=0}^n$ is a sequence of matrix-valued functions which belong to the right (respectively, left) $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$, we associate then the (nonnegative Hermitian) matrix

$$\mathbf{G}_{X,n}^{(F)} := \left(\int_{\mathbb{T}} \underline{X_j}^* \, \mathrm{d}F \underline{X_k}\right)_{j,k=0}^n \tag{8}$$

(respectively, the (nonnegative Hermitian) matrix

$$\mathbf{H}_{X,n}^{(F)} := \left(\int_{\mathbb{T}} \underline{X_j} \, \mathrm{d}F \underline{X_k}^*\right)_{j,k=0}^n\Bigg) \tag{9}$$

with this sequence $(X_j)_{j=0}^n$. In particular, in the case $n = 0$ we have

$$\mathbf{G}_{X,0}^{(F)} = X^*(0)F(\mathbb{T})X(0) \quad \text{and} \quad \mathbf{H}_{X,0}^{(F)} = X(0)F(\mathbb{T})X^*(0). \tag{10}$$

2. A moment problem for rational matrix-valued functions

In this section, we will have a first look on a moment problem for rational matrix-valued functions. This problem consists of the following:

Problem (R). Let $n \in \mathbb{N}$, let $(\alpha_j)_{j=1}^n$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, let $\mathbf{G}$ be a complex $(n+1)q \times (n+1)q$ matrix, and let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$. Describe the set $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$ of all $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that $\mathbf{G}_{X,n}^{(F)} = \mathbf{G}$. In particular, characterize the case $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right] \neq \emptyset$.

Problem (R) is a generalization of the trigonometric moment problem for matrix polynomials. Indeed, if $\alpha_j = 0$ for each $j \in \mathbb{N}_{1,n}$, then $\mathscr{R}_{\alpha,n}^{q\times q}$ coincides with the set of all matrix polynomials of degree not greater than n, in view of Remark 1 the system $X_0 := E_{0,q}, X_1 := E_{1,q}, \ldots, X_n := E_{n,q}$ given in (5) is a basis of the right

$\mathbb{C}^{q\times q}$-module $\mathscr{R}^{q\times q}_{\alpha,n}$, and in view of (1)–(3) the equation $\mathbf{G}^{(F)}_{X,n} = \mathbf{T}^{(F)}_n$ is valid for each $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Hence if $\mathscr{M}\big[(0)^n_{j=1}, \mathbf{G}, (E_k)^n_{k=0}\big] \neq \emptyset$, then $\mathbf{G}$ is necessarily a nonnegative Hermitian block Toeplitz matrix (with $q \times q$ block structure). If $\mathbf{G}$ is a nonnegative Hermitian block Toeplitz matrix (with $q \times q$ block structure), then the set $\mathscr{M}[(0)^n_{j=1}, \mathbf{G}, (E_k)^n_{k=0}]$ is nonempty (see, e.g., [7, Theorem 3.4.2]) and it consists of all $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ which satisfy

$$\int_{\mathbb{T}} z^{-j} F(\mathrm{d}z) = \mathbf{G}_j \tag{11}$$

for each $j \in \mathbb{N}_{0,n}$, where $\mathbf{G} = (\mathbf{G}_{j-k})^n_{j,k=0}$ is the $q \times q$ block partition of $\mathbf{G}$.

In another sense, Problem (R) can be also considered as a generalization of the classical Carathéodory–Nevanlinna–Pick interpolation problem. This can be stated using matrix-valued Carathéodory functions. A function $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ which is holomorphic in $\mathbb{D}$ and which has nonnegative Hermitian real part $\operatorname{Re} \Omega(z)$ for each $z \in \mathbb{D}$ is said to be a $q \times q$ Carathéodory function (in $\mathbb{D}$). The class $\mathscr{C}_q(\mathbb{D})$ of all $q \times q$ Carathéodory functions is intimately connected to the set $\mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. The matricial version of a famous theorem due to G. Herglotz and F. Riesz (see, e.g., [7, Theorem 2.2.2]) shows that, if $\Omega \in \mathscr{C}_q(\mathbb{D})$ is given, there is a unique $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that

$$\Omega(w) - \mathrm{i}\,\mathrm{Im}\Omega(0) = \int_{\mathbb{T}} \frac{z+w}{z-w} F(\mathrm{d}z) \tag{12}$$

for each $w \in \mathbb{D}$. Conversely, if $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is given, then $\Omega_F : \mathbb{D} \to \mathbb{C}^{q\times q}$ defined by

$$\Omega_F(w) := \int_{\mathbb{T}} \frac{z+w}{z-w} F(\mathrm{d}z) \tag{13}$$

belongs to $\mathscr{C}_q(\mathbb{D})$. Let $(\alpha_j)^n_{j=0}$ be a sequence of pairwise different numbers which belong to the unit disk $\mathbb{D}$ where $\alpha_0 = 0$. Further, for $j \in \mathbb{N}_{0,n}$, let ρ_{α_j} be the polynomial defined by $\rho_{\alpha_j}(z) := 1 - \overline{\alpha_j} z$ for each $z \in \mathbb{C}$. Then $\left\{\frac{1}{\rho_{\alpha_j}} \mathbf{I}_q : j \in \mathbb{N}_{0,n}\right\}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}^{q\times q}_{\alpha,n}$. Setting $X_j := \frac{1}{\rho_{\alpha_j}} \mathbf{I}_q$ for each $j \in \mathbb{N}_{0,n}$, we have

$$\mathbf{G}^{(F)}_{X,n} = \left(\frac{1}{2(1-\alpha_j \overline{\alpha_k})} \big(\Omega_F(\alpha_j) + \Omega_F^*(\alpha_k) \big) \right)^n_{j,k=0}$$

for each $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ where $\Omega_F : \mathbb{D} \to \mathbb{C}^{q\times q}$ is given by (13). Hence if there is an $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ which satisfies $\mathbf{G}^{(F)}_{X,n} = \mathbf{G}$, then $\mathbf{G}$ is necessarily a nonnegative Hermitian block Pick matrix of Carathéodory type. Conversely, if $\mathbf{G}$ admits the representation

$$\mathbf{G} = \left(\frac{1}{2(1-\alpha_j \overline{\alpha_k})} (\mathbf{A}_j + \mathbf{A}_k^*) \right)^n_{j,k=0}$$

with some sequence $(\mathbf{A}_j)_{j=0}^n$ of complex $q \times q$ matrices such that $\mathbf{G}$ is a nonnegative Hermitian matrix then one can show that there is an $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that $\mathbf{G}_{X,n}^{(F)} = \mathbf{G}$ (see (12) and [6, Theorem 2]).

Let us mention that in the case $n = 0$ the problem which corresponds to Problem (R) is trivial (see (10)). Further, observe that one can consider a problem analogous to Problem (R) in the left $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$:

Problem (L). Let $n \in \mathbb{N}$, let $(\alpha_j)_{j=1}^n$ be a sequence of numbers which belong to $\mathbb{C} \setminus \mathbb{T}$, let $\mathbf{H}$ be a complex $(n+1)q \times (n+1)q$ matrix, and let $X_0, X_1, \ldots, X_n$ be a basis of the left $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$. Describe the set of all $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that $\mathbf{H}_{X,n}^{(F)} = \mathbf{H}$.

Let us state in which way Problem (L) can be considered as a problem of the type of Problem (R). For $n \in \mathbb{N}_0$, let

$$\tilde{\mathbf{G}}_n^{(\alpha,F)} := \int_{\mathbb{T}} \left(\frac{1}{\pi_{\alpha,n}} e_n^{(q)} \right)^* \mathrm{d}F \left(\frac{1}{\pi_{\alpha,n}} e_n^{(q)} \right) \tag{14}$$

and

$$\tilde{\mathbf{H}}_n^{(\alpha,F)} := \int_{\mathbb{T}} \left(\frac{1}{\pi_{\alpha,n}} d_n^{(q)} \right) \mathrm{d}F \left(\frac{1}{\pi_{\alpha,n}} d_n^{(q)} \right)^* \tag{15}$$

where

$$e_n^{(q)} := (E_{0,q}, E_{1,q}, \ldots, E_{n,q}) \quad \text{and} \quad d_n^{(q)} := \begin{pmatrix} E_{0,q} \\ E_{1,q} \\ \vdots \\ E_{n,q} \end{pmatrix}. \tag{16}$$

In particular, we have

$$\tilde{\mathbf{G}}_0^{(\alpha,F)} = F(\mathbb{T}) \quad \text{and} \quad \tilde{\mathbf{H}}_0^{(\alpha,F)} = F(\mathbb{T}). \tag{17}$$

Note that if F belongs to $\mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, then F^{T} belongs to $\mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ as well.

Remark 2. Let $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence of numbers belonging to $\mathbb{C} \setminus \mathbb{T}$, and let $n \in \mathbb{N}_0$. Then it is readily checked that $\tilde{\mathbf{H}}_n^{(\alpha,F)} = \big(\tilde{\mathbf{G}}_n^{(\alpha,F^{\mathrm{T}})}\big)^{\mathrm{T}}$.

The following remark shows now especially the interrelation between Problems (R) and (L).

Remark 3. Let F_1 and F_2 belong to $\mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence of numbers belonging to $\mathbb{C} \setminus \mathbb{T}$, and let $n \in \mathbb{N}_0$. In view of Remarks 1 and 2, one can easily see that the following statements are equivalent:

(i) For every choice of X in $\mathscr{R}_{\alpha,n}^{q \times p}$ and Y in $\mathscr{R}_{\alpha,n}^{q \times r}$, $\int_{\mathbb{T}} \underline{X}^* \, \mathrm{d}F_1 \underline{Y} = \int_{\mathbb{T}} \underline{X}^* \, \mathrm{d}F_2 \underline{Y}$.

(ii) There is a basis $X_0, X_1, \dots, X_n$ of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$ such that $\mathbf{G}_{X,n}^{(F_1)} = \mathbf{G}_{X,n}^{(F_2)}$.

(iii) For every choice of X in $\mathscr{R}_{\alpha,n}^{p\times q}$ and Y in $\mathscr{R}_{\alpha,n}^{r\times q}$, $\int_{\mathbb{T}} \underline{X}\, \mathrm{d}F_1 \underline{Y}^* = \int_{\mathbb{T}} \underline{X}\, \mathrm{d}F_2 \underline{Y}^*$.

(iv) There is a basis $X_0, X_1, \dots, X_n$ of the left $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$ such that $\mathbf{H}_{X,n}^{(F_1)} = \mathbf{H}_{X,n}^{(F_2)}$.

Because of Remark 3 we focus our attention to Problem (R). The next considerations are aimed at characterizing the case $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right] \neq \emptyset$.

Remark 4. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, let $n \in \mathbb{N}_0$, and let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$. Then one can verify that there is a unique complex $(n+1)q \times (n+1)q$ matrix $\mathbf{R}$ such that

$$\frac{1}{\pi_{\alpha,n}} e_n^{(q)} = X^{[n]}\mathbf{R} \tag{18}$$

where $e_n^{(q)}$ is given by (16) and where

$$X^{[n]} := (X_0, X_1, \dots, X_n). \tag{19}$$

This matrix $\mathbf{R}$ is nonsingular and satisfies

$$\tilde{\mathbf{G}}_n^{(\alpha,F)} = \mathbf{R}^* \mathbf{G}_{X,n}^{(F)} \mathbf{R}$$

for each $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

In order to give a characterization of the case $\mathscr{M}\left[(\alpha_j)_{j=0}^n, \mathbf{G}; (X_k)_{k=0}^n\right] \neq \emptyset$, it seems to be useful to associate particular nonnegative Hermitian Borel measures with a given measure $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. If $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, $n \in \mathbb{N}$ and a sequence $(\alpha_j)_{j=1}^n$ of numbers belonging to $\mathbb{C}\setminus\mathbb{T}$ are given, then it is readily checked that $F^{(\alpha,n)} : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q\times q}$ and $F_{(\alpha,n)} : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q\times q}$ defined by

$$F^{(\alpha,n)}(B) := \int_B \left(\underline{\frac{1}{\pi_{\alpha,n}} \mathbf{I}_q}\right)^* \mathrm{d}F \left(\underline{\frac{1}{\pi_{\alpha,n}} \mathbf{I}_q}\right) \tag{20}$$

and

$$F_{(\alpha,n)}(B) := \int_B \left(\underline{\pi_{\alpha,n}\, \mathbf{I}_q}\right)^* \mathrm{d}F \left(\underline{\pi_{\alpha,n}\, \mathbf{I}_q}\right) \tag{21}$$

belong both to $\mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ as well and the identities

$$(F^{(\alpha,n)})_{(\alpha,n)} = F \quad \text{and} \quad (F_{(\alpha,n)})^{(\alpha,n)} = F \tag{22}$$

are satisfied (see also [9, Lemma 3.1]).

Proposition 5. *Let $n \in \mathbb{N}$, let $(\alpha_j)_{j=1}^n$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, and let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$. Let $\mathbf{R}$ be* *the unique complex* $(n+1)q \times (n+1)q$ *matrix such that* (18) *holds. Further, let* $\mathbf{G}$ *be a complex* $(n+1)q \times (n+1)q$ *matrix and let* $\mathbf{\Gamma} := \mathbf{R}^*\mathbf{G}\mathbf{R}$.

(a) *The set* $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$ *is nonempty if and only if* $\mathbf{\Gamma}$ *is a nonnegative Hermitian* $q \times q$ *block Toeplitz matrix.*
(b) *Let* $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Then* F *belongs to* $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$ *if and only if* $F^{(\alpha,n)}$ *belongs to* $\mathscr{M}\left[(0)_{j=1}^n, \mathbf{\Gamma}, (E_k)_{k=0}^n\right]$.

Proof. First assume that $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$ is nonempty. Let $F \in \mathscr{M}\left[(\alpha_j)_{j=0}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$. Using (20) and (18) we get then

$$\mathbf{T}_n^{(F^{(\alpha,n)})} = \left(\int_{\mathbb{T}} E_j^* \mathrm{d}F^{(\alpha,n)} E_k\right)_{j,k=0}^n = \int_{\mathbb{T}} \left(\frac{1}{\pi_{\alpha,n}} e_n^{(q)}\right)^* \mathrm{d}F \left(\frac{1}{\pi_{\alpha,n}} e_n^{(q)}\right)$$
$$= \int_{\mathbb{T}} (X^{[n]}\mathbf{R})^* \mathrm{d}F (X^{[n]}\mathbf{R}) = \mathbf{R}^* \mathbf{G}_{X,n}^{(F)} \mathbf{R} = \mathbf{R}^*\mathbf{G}\mathbf{R} = \mathbf{\Gamma}.$$

Consequently, $\mathbf{\Gamma}$ is a nonnegative Hermitian block Toeplitz matrix (with $q \times q$ block structure) and $F^{(\alpha,n)}$ belongs to $\mathscr{M}[(0)_{j=1}^n, \mathbf{\Gamma}, (E_k)_{k=0}^n]$. Conversely, now assume that $\mathbf{\Gamma}$ is a nonnegative Hermitian block Toeplitz matrix (with $q \times q$ block structure). Then there is an $M \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that

$$\left(\int_{\mathbb{T}} E_j^* \,\mathrm{d}M E_k\right)_{j,k=0}^n = \mathbf{\Gamma} \tag{23}$$

(see, e.g., [7, Theorem 3.4.2]). In view of (21) and Remark 4, we obtain

$$\mathbf{\Gamma} = \int_{\mathbb{T}} (\underline{e_n^{(q)}})^* \mathrm{d}M \underline{e_n^{(q)}} = \int_{\mathbb{T}} \left(\underline{\frac{1}{\pi_{\alpha,n}} e_n^{(q)}}\right)^* \mathrm{d}M_{(\alpha,n)} \left(\underline{\frac{1}{\pi_{\alpha,n}} e_n^{(q)}}\right)$$
$$= \int_{\mathbb{T}} (\underline{X^{[n]}}\mathbf{R})^* \mathrm{d}M_{(\alpha,n)} (\underline{X^{[n]}}\mathbf{R}) = \mathbf{R}^* \mathbf{G}_{X,n}^{(M_{(\alpha,n)})} \mathbf{R} \tag{24}$$

and therefore $\mathbf{G}_{X,n}^{(M_{(\alpha,n)})} = \mathbf{G}$. Hence $M_{(\alpha,n)}$ belongs to $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$. Now we assume that $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is such that the measure $F^{(\alpha,n)}$ belongs to $\mathscr{M}\left[(0)_{j=1}^n, \mathbf{\Gamma}, (E_k)_{k=0}^n\right]$. Then $M := F^{(\alpha,n)}$ satisfies (23). From (22) and (24) it follows $F = M_{(\alpha,n)}$ and

$$\mathbf{G}_{X,n}^{(F)} = \mathbf{R}^{-*}\mathbf{\Gamma}\mathbf{R}^{-1} = \mathbf{G},$$

i.e., F belongs to $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$. $\square$

Note that, at the end of Section 5 we will again turn our attention to the set $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right]$. There, in the case that $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n\right] \neq \emptyset$ and the given matrix $\mathbf{G}$ is positive Hermitian, we will construct a concrete measure which belongs to this set. In a forthcoming paper, we want to give a description of $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}, (X_k)_{k=0}^n\right]$ with the aid of matricial Carathéodory functions. More precisely, we want to describe the set of all Riesz–Herglotz transforms of the set $\mathcal{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}, (X_k)_{k=0}^n\right]$.

3. On reproducing kernels associated with nondegenerate nonnegative Hermitian Borel measures

Let $n \in \mathbb{N}_0$. A nonnegative Hermitian measure $F \in \mathcal{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is called nondegenerate of order n if the block Toeplitz matrix $\mathbf{T}_n^{(F)}$ given by (1) and (3) is nonsingular. We will use $\mathcal{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ to denote the set of all $F \in \mathcal{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ which are nondegenerate of order n. Further, let

$$\mathcal{M}_{\geqslant}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) := \bigcap_{m=0}^{\infty} \mathcal{M}_{\geqslant}^{q,m}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}).$$

The set $\mathcal{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ can be characterized in the following way.

Theorem 6. *Let $F \in \mathcal{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and let $n \in \mathbb{N}_0$. Then the following statements are equivalent*:

(i) *F is nondegenerate of order n.*
(ii) *There exist a sequence $(\alpha_j)_{j\in\mathbb{N}}$ from $\mathbb{C}\setminus\mathbb{T}$ and a basis $(X_k)_{k=0}^n$ of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ such that the matrix $\mathbf{G}_{X,n}^{(F)}$ given by* (8) *is nonsingular.*
(iii) *For every sequence $(\alpha_j)_{j\in\mathbb{N}}$ from $\mathbb{C}\setminus\mathbb{T}$ and every basis $(X_k)_{k=0}^n$ of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, the matrix $\mathbf{G}_{X,n}^{(F)}$ given by* (8) *is positive Hermitian.*
(iv) *There exist a sequence $(\alpha_j)_{j\in\mathbb{N}}$ from $\mathbb{C}\setminus\mathbb{T}$ and a basis $(X_k)_{k=0}^n$ of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ such that the matrix $\mathbf{H}_{X,n}^{(F)}$ given by* (9) *is nonsingular.*
(v) *For every sequence $(\alpha_j)_{j\in\mathbb{N}}$ from $\mathbb{C}\setminus\mathbb{T}$ and every basis $(X_k)_{k=0}^n$ of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, the matrix $\mathbf{H}_{X,n}^{(F)}$ given by* (9) *is positive Hermitian.*

A proof of Theorem 6 is given in [9, Theorem 5.6].

Remark 7. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, let $n \in \mathbb{N}_0$, let $F \in \mathcal{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and let $Z \in \mathcal{R}_{\alpha,n}^{q\times p}$. From Remark 1 one can then easily see that there is a complex $(n+1)q \times p$ matrix $\mathbf{Q}$ such that

$$Z = \frac{1}{\pi_{\alpha,n}} e_n^{(q)} \mathbf{Q}$$

which implies

$$\int_{\mathbb{T}} \underline{Z}^* \mathrm{d}F \underline{Z} = \left(\sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}\mathbf{Q}\right)^* \sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}\mathbf{Q}. \tag{25}$$

Hence Theorem 6 shows that the left-hand side of (25) coincides with $\mathbf{0}_{p\times p}$ if and only if $Z = O_{q\times p}$.

Now let us consider a sequence $(\alpha_j)_{j\in\mathbb{N}}$ of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$. Let $n \in \mathbb{N}_0$ and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. In view of (6), (7) and Theorem 6 one can easily see that $\left(\mathscr{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_r\right)$ is a right $\mathbb{C}^{q\times q}$-Hilbert module and that $\left(\mathscr{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_l\right)$ is a left $\mathbb{C}^{q\times q}$-Hilbert module. Moreover, we will check that $K_{n;r}^{(\alpha,F)} : (\mathbb{C}_0\backslash P_{\alpha,n}) \times (\mathbb{C}_0\backslash P_{\alpha,n}) \to \mathbb{C}^{q\times q}$ given by (4), (5), (14), (16) and

$$K_{n;r}^{(\alpha,F)}(v, w) := \left(\left[\frac{1}{\pi_{\alpha,n}} e_n^{(q)}\right](v)\right) (\tilde{\mathbf{G}}_n^{(\alpha,F)})^{-1} \left(\left[\frac{1}{\pi_{\alpha,n}} e_n^{(q)}\right](w)\right)^* \tag{26}$$

(respectively, $K_{n;l}^{(\alpha,F)} : (\mathbb{C}_0\backslash P_{\alpha,n}) \times (\mathbb{C}_0\backslash P_{\alpha,n}) \to \mathbb{C}^{q\times q}$ given by (4), (5), (15), (16) and

$$K_{n;l}^{(\alpha,F)}(v, w) := \left(\left[\frac{1}{\pi_{\alpha,n}} d_n^{(q)}\right](v)\right)^* (\tilde{\mathbf{H}}_n^{(\alpha,F)})^{-1} \left(\left[\frac{1}{\pi_{\alpha,n}} d_n^{(q)}\right](w)\right)\Bigg) \tag{27}$$

is a reproducing kernel of the right $\mathbb{C}^{q\times q}$-Hilbert module $\left(\mathscr{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_r\right)$ (respectively, in the left $\mathbb{C}^{q\times q}$-Hilbert module $\left(\mathscr{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_l\right)$ (compare Theorem 10). For this reason, it seems to be useful to introduce further notations. If $v \in \mathbb{C}_0\backslash P_{\alpha,n}$ is given, then, in particular, we will use the mappings $A_{n,v}^{(\alpha,F)} : \mathbb{C}_0\backslash P_{\alpha,n} \to \mathbb{C}^{q\times q}$ and $B_{n,v}^{(\alpha,F)} : \mathbb{C}_0\backslash P_{\alpha,n} \to \mathbb{C}^{q\times q}$ given by

$$A_{n,v}^{(\alpha,F)}(w) := K_{n;r}^{(\alpha,F)}(w, v) \quad \text{and} \quad B_{n,v}^{(\alpha,F)}(w) := K_{n;l}^{(\alpha,F)}(v, w). \tag{28}$$

The matrix-valued functions $A_{n,v}^{(\alpha,F)}$ and $B_{n,v}^{(\alpha,F)}$ both belong to $\mathscr{R}_{\alpha,n}^{q\times q}$. In particular, from (17) we see $A_{0,v}^{(\alpha,F)}$ and $B_{0,v}^{(\alpha,F)}$ coincide both with the constant matrix-valued function with value $(F(\mathbb{T}))^{-1}$. The following remark shows that we can essentially concentrate our attention to the right $\mathbb{C}^{q\times q}$-Hilbert module $\left(\mathscr{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_r\right)$.

Remark 8. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence from $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. In view of Remark 2 and Theorem 6 one can then easily see that F^{T} belongs to $\mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ as well and that $K_{n;l}^{(\alpha,F)}(v, w) = \left(K_{n;r}^{(\alpha,F^{\mathrm{T}})}(w, v)\right)^{\mathrm{T}}$ for every choice of v and w in $\mathbb{C}_0\backslash P_{\alpha,n}$. In particular, $B_{n,v}^{(\alpha,F)} = \left(A_{n,v}^{(\alpha,F^{\mathrm{T}})}\right)^{\mathrm{T}}$ for each $v \in \mathbb{C}_0\backslash P_{\alpha,n}$.

The rational matrix-valued function $A_{n,v}^{(\alpha,F)}$ and $B_{n,v}^{(\alpha,F)}$ are generalizations of certain matrix polynomials which play a key role in the theory of orthogonal matrix polynomials corresponding to a nondegenerate nonnegative Hermitian measure (see [4] and [7, Section 3.6]). Let us turn our attention to this interrelation to the polynomial case. If $n \in \mathbb{N}_0$ and if $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, then we consider the matrix polynomials $A_n^{[F]}$ and $B_n^{[F]}$ given by

$$\begin{aligned} &A_n^{[F]}(z) := e_n^{(q)}(z)\left(\mathbf{T}_n^{(F)}\right)^{-1}\left(e_n^{(q)}(0)\right)^* \quad \text{and} \\ &B_n^{[F]}(z) := \left(\varepsilon_n^{(q)}(0)\right)^*\left(\mathbf{T}_n^{(F)}\right)^{-1}\varepsilon_n^{(q)}(z) \end{aligned} \tag{29}$$

for each $z \in \mathbb{C}$, where the matrix polynomials $e_n^{(q)}$ and $\varepsilon_n^{(q)}$ are given by (16) and

$$\varepsilon_n^{(q)} := (E_{n,q}, E_{n-1,q}, \ldots, E_{0,q})^{\mathrm{T}}. \tag{30}$$

Remark 9. Let $n \in \mathbb{N}_0$, let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$ such that $\alpha_j = 0$ for each integer j satisfying $1 \leqslant j \leqslant n$. Then it is readily verified that $A_{n,0}^{(\alpha,F)} = A_n^{[F]}$ and $B_{n,0}^{(\alpha,F)} = B_n^{[F]}$.

Theorem 10. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers belonging to $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Further let $v \in \mathbb{C}_0\backslash P_{\alpha,n}$.*

(a) *There is a unique $Y \in \mathscr{R}_{\alpha,n}^{q\times q}$ such that*

$$\int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{X} = X(v) \tag{31}$$

for each $X \in \mathscr{R}_{\alpha,n}^{q\times p}$, namely $Y = A_{n,v}^{(\alpha,F)}$.

(b) *There is a unique $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$ such that*

$$\int_{\mathbb{T}} \underline{X} \mathrm{d}F \underline{Z}^* = X(v) \tag{32}$$

for all $X \in \mathscr{R}_{\alpha,n}^{p\times q}$, namely $Z = B_{n,v}^{(\alpha,F)}$.

Proof. (a) Let $X \in \mathscr{R}_{\alpha,n}^{q\times p}$. In view of Remark 1, there is a unique complex matrix $\mathbf{C}$ such that $X = \frac{1}{\pi_{\alpha,n}} e_n^{(q)} \mathbf{C}$. Therefore, from (28) and (14) we see

$$\begin{aligned} &\int_{\mathbb{T}} \underline{\left(A_{n,v}^{(\alpha,F)}\right)^*} \mathrm{d}F \underline{X} \\ &= \left[\frac{1}{\pi_{\alpha,n}} e_n^{(q)}\right](v) \left(\tilde{\mathbf{G}}_n^{(\alpha,F)}\right)^{-1} \left(\int_{\mathbb{T}} \underline{\left(\frac{1}{\pi_{\alpha,n}} e_n^{(q)}\right)^*} \mathrm{d}F \underline{\frac{1}{\pi_{\alpha,n}} e_n^{(q)}}\right) \mathbf{C} \\ &= X(v). \end{aligned} \tag{33}$$

Now let $Y \in \mathscr{R}_{\alpha,n}^{q\times q}$ be arbitrary such that (31) holds for every choice of X in $\mathscr{R}_{\alpha,n}^{q\times p}$. Let us consider an arbitrary complex $q \times p$ matrix $\mathbf{X}$. Then the matrix-valued functions $S := Y\mathbf{X}$ and $T := A_{n,v}^{(\alpha,F)}\mathbf{X}$ both belong to $\mathscr{R}_{\alpha,n}^{q\times p}$ and the considerations above show that

$$\int_{\mathbb{T}} \underline{\left(A_{n,v}^{(\alpha,F)}\right)^*} \mathrm{d}F \underline{S} = S(v) = Y(v)\mathbf{X} \quad \text{and}$$

$$\int_{\mathbb{T}} \underline{\left(A_{n,v}^{(\alpha,F)}\right)^*} \mathrm{d}F \underline{T} = T(v) = A_{n,v}^{(\alpha,F)}(v)\mathbf{X}.$$

On the other hand, by assumption we have

$$\int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{S} = S(v) = Y(v)\mathbf{X} \quad \text{and} \quad \int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{T} = T(v) = A_{n,v}^{(\alpha,F)}(v)\mathbf{X}.$$

Consequently, it follows

$$\left[\int_{\mathbb{T}} \big(\underline{Y} - \underline{A_{n,v}^{(\alpha,F)}}\big)^* \mathrm{d}F \big(\underline{Y} - \underline{A_{n,v}^{(\alpha,F)}}\big)\right] \mathbf{X}$$
$$= \int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{S} - \int_{\mathbb{T}} \big(\underline{A_{n,v}^{(\alpha,F)}}\big)^* \mathrm{d}F \underline{S} - \int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{T} + \int_{\mathbb{T}} \big(\underline{A_{n,v}^{(\alpha,F)}}\big)^* \mathrm{d}F \underline{T} = \mathbf{0}.$$

This implies

$$\int_{\mathbb{T}} \big(\underline{Y} - \underline{A_{n,v}^{(\alpha,F)}}\big)^* \mathrm{d}F \big(\underline{Y} - \underline{A_{n,v}^{(\alpha,F)}}\big) = \mathbf{0}$$

and, in view of Remark 7, therefore $Y = A_{n,v}^{(\alpha,F)}$.

(b) From Remark 2 and Theorem 6 we see that F^{T} belongs to $\mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Thus we obtain from part (a) that there is a unique $W \in \mathscr{R}_{\alpha,n}^{q\times q}$ such that

$$\int_{\mathbb{T}} \underline{W}^* \mathrm{d}F^{\mathrm{T}} \underline{X}^{\mathrm{T}} = X^{\mathrm{T}}(v)$$

for all $X \in \mathscr{R}_{\alpha,n}^{p\times q}$, namely $W = A_{n,v}^{(\alpha,F^{\mathrm{T}})}$. In view of Remark 8 we see then that there exists a unique $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$ such that (32) is satisfied for each $X \in \mathscr{R}_{\alpha,n}^{p\times q}$, namely $Z = B_{n,v}^{(\alpha,F)}$. □

Corollary 11. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}$, and $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Further, let v and w belong to $\mathbb{C}_0\backslash P_{\alpha,n}$. Then $v = w$ if and only if there is a $\mathbf{C} \in \mathbb{C}^{q\times q}$ such that $A_{n,v}^{(\alpha,F)} = A_{n,w}^{(\alpha,F)}\mathbf{C}$.*

Proof. Clearly, $v = w$ implies $A_{n,v}^{(\alpha,F)} = A_{n,w}^{(\alpha,F)}\mathbf{I}_q$. Conversely, suppose that $A_{n,v}^{(\alpha,F)} = A_{n,w}^{(\alpha,F)}\mathbf{C}$ with some $\mathbf{C} \in \mathbb{C}^{q\times q}$. Let $f_{w,q} : \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by $f_{w,q}(u) := (u - w)\mathbf{I}_q$. Then $X := \frac{1}{\pi_{\alpha,n}} f_{\omega,q}$ belongs obviously to $\mathscr{R}_{\alpha,n}^{q\times q}$ and satisfies $X(w) = \mathbf{0}$. From Theorem 10 we see that

$$X(v) = \int_{\mathbb{T}} \big(\underline{A_{n,v}^{(\alpha,F)}}\big)^* \mathrm{d}F \underline{X} = \mathbf{C}^* \int_{\mathbb{T}} \big(\underline{A_{n,w}^{(\alpha,F)}}\big)^* \mathrm{d}F \underline{X} = \mathbf{C}^* X(w) = \mathbf{0}.$$

Since $X(v) = \mathbf{0}$ implies $v = w$, the proof is finished. □

At the end of this section we give some remarks on the reproducing kernels considered above.

Remark 12. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence from $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, let $(X_k)_{k=0}^n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$, and let $X^{[n]}$ be given by (19). Further, let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Using Remark 4 and (26) it is then readily checked that

$$K_{n;r}^{(\alpha,F)}(v, w) = X^{[n]}(v)\big(\mathbf{G}_{X,n}^{(F)}\big)^{-1}(X^{[n]}(w))^*$$

holds for every choice of v and w in $\mathbb{C}_0 \setminus P_{\alpha,n}$. In particular, if $v \in \mathbb{C}_0 \setminus P_{\alpha,n}$ is given, then

$$A_{n,v}^{(\alpha,F)}(w) = X^{[n]}(w)\big(\mathbf{G}_{X,n}^{(F)}\big)^{-1}(X^{[n]}(v))^*$$

for each $w \in \mathbb{C}_0 \setminus P_{\alpha,n}$.

Analogously, one can obtain a left variant of Remark 12. We omit the details.

Remark 13. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, let $n \in \mathbb{N}_0$, let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and let $(v_k)_{k=0}^n$ be a sequence of pairwise different points which belong to $\mathbb{C}_0 \setminus P_{\alpha,n}$. The application of Theorem 10 shows that

$$\int_{\mathbb{T}} \big(A_{n,v_j}^{(\alpha,F)}\big)^* \mathrm{d}F A_{n,v_k}^{(\alpha,F)} = A_{n,v_k}^{(\alpha,F)}(v_j) = K_{n;r}^{(\alpha,F)}(v_j, v_k) \tag{34}$$

holds for every choice of j and k in $\mathbb{N}_{0,n}$. Moreover, one can easily see that $\big\{A_{n,v_j}^{(\alpha,F)} : j \in \mathbb{N}_{0,n}\big\}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$.

Remark 14. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, let $n, m \in \mathbb{N}_0$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Further, let $(v_k)_{k=0}^n$ be a sequence of pairwise different points which belong to $\mathbb{C}_0 \setminus P_{\alpha,n}$. Then in view of Remark 13 it is easily checked that $\big(K_{n;r}^{(\alpha,F)}(v_j, v_k)\big)_{j,k=0}^m$ is positive Hermitian if and only if $m \leqslant n$.

4. Extremal problems in the space $\mathscr{R}_{\alpha,n}^{q\times p}$

This section is aimed at considering rational matricial generalizations of extremal problems which were partially studied by Grenander and Szegö [10] in the classical polynomial case, by Bultheel et al. [3] in the scalar rational case, by Delsarte et al. [4] in the case of matrix polynomials, and by Bultheel [2] in general Hilbert modules. Hereby, we restrict our consideration to the right variants of these questions. The left ones can be discussed analogously. We start with a generalization of a problem which plays a key role for the foundation of the theory of orthogonal matrix polynomials laid in [4].

Remark 15. Let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, and let $n \in \mathbb{N}_0$. For each $v \in \mathbb{C}_0 \setminus P_{\alpha,n}$, let $\varphi_{v,n}^{(\alpha,F)} : \mathscr{R}_{\alpha,n}^{q\times q} \to \mathbb{C}^{q\times q}$ (respectively, $\psi_{v,n}^{(\alpha,F)} : \mathscr{R}_{\alpha,n}^{q\times q} \to \mathbb{C}^{q\times q}$) be defined by

$$\varphi_{v,n}^{(\alpha,F)}(Z) := \int_{\mathbb{T}} \underline{Z}^* \mathrm{d}F \underline{Z} - 2\,\mathrm{Re}\,Z(v) \tag{35}$$

$$\left(\text{respectively, } \psi_{v,n}^{(\alpha,F)}(Z) := \int_{\mathbb{T}} \underline{Z}\,\mathrm{d}F \underline{Z}^* - 2\,\mathrm{Re}\,Z(v)\right). \tag{36}$$

Obviously, if $m \in \mathbb{N}_0$ satisfies $m \geqslant n$, then, for each $v \in \mathbb{C}_0 \backslash P_{\alpha,m}$, we have

$$\varphi_{v,n}^{(\alpha,F)} = \text{Rstr.}_{\mathscr{R}_{\alpha,n}^{q\times q}} \varphi_{v,m}^{(\alpha,F)} \quad \left(\text{respectively, } \psi_{v,n}^{(\alpha,F)} = \text{Rstr.}_{\mathscr{R}_{\alpha,n}^{q\times q}} \psi_{v,m}^{(\alpha,F)}\right).$$

The matrix-valued functions given by (35) and (36) can be considered as generalizations of the functions which are used by Delsarte et al. [4] in the matrix polynomial case ($\alpha_j = 0$ for each $j \in \mathbb{N}$ and $v = 0$).

Note that if $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$, then $Z^{\mathrm{T}} \in \mathscr{R}_{\alpha,n}^{q\times q}$. Thus the following remark shows that we can restrict our attention to the functions given by (35).

Remark 16. Let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence from $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, and let $v \in \mathbb{C}_0\backslash P_{\alpha,n}$. For each $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$, then

$$\psi_{v,n}^{(\alpha,F)}(Z) = \left(\varphi_{v,n}^{(\alpha,F^{\mathrm{T}})}(Z^{\mathrm{T}})\right)^{\mathrm{T}}.$$

Remark 17. Let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence from $\mathbb{C}\backslash\mathbb{T}$, and let $n \in \mathbb{N}_0$. Further, let $v \in \mathbb{C}_0\backslash P_{\alpha,n}$ and let $\{X_0, X_1, \ldots, X_n\}$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$. For each $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$, then

$$\varphi_{v,n}^{(\alpha,F)}(Z) = \mathbf{C}^*\mathbf{G}_{X,n}^{(F)}\mathbf{C} - 2\,\mathrm{Re}Z(v)$$

where $\mathbf{C}$ is the unique complex $(n+1)q \times q$ matrix such that $Z = X^{[n]}\mathbf{C}$.

Theorem 18. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers belonging to $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and let $v \in \mathbb{C}_0\backslash P_{\alpha,n}$. Then there is a unique $X \in \mathscr{R}_{\alpha,n}^{q\times q}$ such that*

$$\varphi_{v,n}^{(\alpha,F)}(Z) \geqslant \varphi_{v,n}^{(\alpha,F)}(X) \tag{37}$$

for each $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$, namely $X = A_{n,v}^{(\alpha,F)}$. Moreover,

$$\varphi_{v,n}^{(\alpha,F)}\left(A_{n,v}^{(\alpha,F)}\right) = -K_{n;r}^{(\alpha,F)}(v, v). \tag{38}$$

Proof. We already observed that $A_{n,v}^{(\alpha,F)}$ belongs to $\mathscr{R}_{\alpha,n}^{q\times q}$. Let $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$. Because of Remark 1 there is a unique complex matrix $\mathbf{C}$ such that

$$Z = \frac{1}{\pi_{\alpha,n}} e_n^{(q)}\mathbf{C}, \tag{39}$$

and in view of Theorem 6, we set

$$\mathbf{D} := \sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}\mathbf{C} - \sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}^{-1}\left(\left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\right)^*.$$

Hence we have

$$\mathbf{C} = \sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}^{-1}\mathbf{D} + \left(\tilde{\mathbf{G}}_n^{(\alpha,F)}\right)^{-1}\left(\left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\right)^*. \tag{40}$$

Using (26) we get then

$$\begin{aligned} 2\,\mathrm{Re}\,Z(v) &= 2\mathrm{Re}\left(\left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\mathbf{C}\right) \\ &= \left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}^{-1}\mathbf{D} + \mathbf{D}^*\sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}^{-1}\left(\left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\right)^* \\ &\quad + 2K_{n;r}^{(\alpha,F)}(v,v) \end{aligned}$$

and

$$\begin{aligned} \mathbf{C}^*\tilde{\mathbf{G}}_n^{(\alpha,F)}\mathbf{C} = \mathbf{D}^*\mathbf{D} + \mathbf{D}^*\sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}^{-1}\left(\left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\right)^* \\ + \left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\sqrt{\tilde{\mathbf{G}}_n^{(\alpha,F)}}^{-1}\mathbf{D} + K_{n;r}^{(\alpha,F)}(v,v). \end{aligned}$$

From Remark 17 it follows then

$$\varphi_{v,n}^{(\alpha,F)}(Z) + K_{n;r}^{(\alpha,F)}(v,v) = \mathbf{D}^*\mathbf{D}. \tag{41}$$

The right-hand side of (41) is obviously nonnegative Hermitian. Hence

$$\varphi_{v,n}^{(\alpha,F)}(Z) \geqslant -K_{n;r}^{(\alpha,F)}(v,v),$$

where equality holds if and only if $\mathbf{D} = \mathbf{0}$, i.e., in view of (40), (39) and (28), if and only if $Z = A_{n,v}^{(\alpha,F)}$. □

Corollary 19. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers belonging to $\mathbb{C}\setminus\mathbb{T}$, let $n \in \mathbb{N}$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Let $v \in \mathbb{C}_0\setminus P_{\alpha,n}$. For each $m \in \mathbb{N}_{0,n-1}$, then*

$$K_{m+1;r}^{(\alpha,F)}(v,v) \geqslant K_{m;r}^{(\alpha,F)}(v,v) \geqslant (F(\mathbb{T}))^{-1}.$$

In particular, $\left(A_{m,v}^{(\alpha,F)}(v)\right)_{m=0}^{n}$ is a nondecreasing sequence of positive Hermitian complex $q \times q$ matrices (with respect to the Löwner semi-ordering).

Proof. Obviously, F belongs to $\bigcap_{m=0}^{n}\mathscr{M}_{\geqslant}^{q,m}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ (see, e.g., [9, Remark 5.3]). In particular, from (26), (17), and Theorem 6 we see

$$K_{0;r}^{(\alpha,F)}(v,v) = \left(\tilde{\mathbf{G}}_0^{(\alpha,F)}\right)^{-1} = (F(\mathbb{T}))^{-1}$$

for all points $v \in \mathbb{C}_0\setminus P_{\alpha,n}$. Let $m \in \mathbb{N}_{0,n-1}$. From $\mathscr{R}_{\alpha,m}^{q\times q} \subseteq \mathscr{R}_{\alpha,m+1}^{q\times q}$ and Theorem 18 the assertion immediately follows. □

Corollary 20. *Let $n \in \mathbb{N}_0$ and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. If $n > 0$, then let $(\alpha_j)_{j=1}^{n}$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$. Further, let $v \in \mathbb{C}_0\setminus P_{\alpha,n}$ and let $X \in \mathscr{R}_{\alpha,n}^{q\times q}$. Then*

$$\int_{\mathbb{T}} \left(\underline{X} - \underline{A_{n,v}^{(\alpha,F)}}\right)^* \mathrm{d}F\left(\underline{X} - \underline{A_{n,v}^{(\alpha,F)}}\right) = \varphi_{v,n}^{(\alpha,F)}(X) - \varphi_{v,n}^{(\alpha,F)}\left(A_{n,v}^{(\alpha,F)}\right). \tag{42}$$

Proof. Because of Theorem 10, Eqs. (33) and

$$\int_{\mathbb{T}} \big(\underline{A_{n,v}^{(\alpha,F)}}\big)^* \mathrm{d}F \underline{A_{n,v}^{(\alpha,F)}} = A_{n,v}^{(\alpha,F)}(v) = K_{n;r}^{(\alpha,F)}(v,v)$$

hold. On the other hand from Theorem 18 we know that (38) is satisfied. Hence, in view of (35), it follows from (42)

$$\begin{aligned}\int_{\mathbb{T}} \big(\underline{X} - \underline{A_{n,v}^{(\alpha,F)}}\big)^* \mathrm{d}F \big(\underline{X} - \underline{A_{n,v}^{(\alpha,F)}}\big) &= \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X} - 2\,\mathrm{Re}X(v) + K_{n;r}^{(\alpha,F)}(v,v) \\ &= \varphi_{v,n}^{(\alpha,F)}(X) - \varphi_{v,n}^{(\alpha,F)}\big(A_{n,v}^{(\alpha,F)}\big). \qquad \square\end{aligned}$$

Corollary 21. *Let n and m belong to $\mathbb{N}_0$ and let $F \in \mathscr{M}_{\geqslant}^{q,n+m}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. If $n+m > 0$, then let $(\alpha_j)_{j=1}^{n+m}$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$. Further, let $v \in \mathbb{C}_0\backslash P_{\alpha,n+m}$. Then $A_{n,v}^{(\alpha,F)}$ is exactly the orthogonal projection of $A_{n+m,v}^{(\alpha,F)}$ onto the right $\mathbb{C}^{q\times q}$-submodule $\mathscr{R}_{\alpha,n}^{q\times q}$ in the right $\mathbb{C}^{q\times q}$-Hilbert module $\big(\mathscr{R}_{\alpha,n+m}^{q\times q}, (\cdot,\cdot)_r\big)$. Moreover, the following statements are equivalent*:

(i) $A_{n,v}^{(\alpha,F)} = A_{n+m,v}^{(\alpha,F)}$.
(ii) $K_{n;r}^{(\alpha,F)}(v,v) = K_{n+m;r}^{(\alpha,F)}(v,v)$.
(iii) $A_{n+m,v}^{(\alpha,F)}$ *belongs to* $\mathscr{R}_{\alpha,n}^{q\times q}$.

Proof. First observe that $F \in \mathscr{M}_{\geqslant}^{q,n+m}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ implies $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ (see [9, Remark 5.3]) and that $A_{n,v}^{(\alpha,F)}$ belongs to $\mathscr{R}_{\alpha,n}^{q\times q}$. Let $X \in \mathscr{R}_{\alpha,n}^{q\times q}$. From (6), Corollary 20 and Remark 15 we see

$$\big(A_{n+m,v}^{(\alpha,F)} - X, A_{n+m,v}^{(\alpha,F)} - X\big)_r = \varphi_{v,n}^{(\alpha,F)}(X) - \varphi_{v,n+m}^{(\alpha,F)}\big(A_{n+m,v}^{(\alpha,F)}\big).$$

In particular,

$$\big(A_{n+m,v}^{(\alpha,F)} - A_{n,v}^{(\alpha,F)}, A_{n+m,v}^{(\alpha,F)} - A_{n,v}^{(\alpha,F)}\big)_r = \varphi_{v,n}^{(\alpha,F)}\big(A_{n,v}^{(\alpha,F)}\big) - \varphi_{v,n+m}^{(\alpha,F)}\big(A_{n+m,v}^{(\alpha,F)}\big). \tag{43}$$

Hence, from Theorem 18 we obtain

$$\begin{aligned}&\big(A_{n+m,v}^{(\alpha,F)} - X, A_{n+m,v}^{(\alpha,F)} - X\big)_r - \big(A_{n+m,v}^{(\alpha,F)} - A_{n,v}^{(\alpha,F)}, A_{n+m,v}^{(\alpha,F)} - A_{n,v}^{(\alpha,F)}\big)_r \\ &\quad = \varphi_{v,n}^{(\alpha,F)}(X) - \varphi_{v,n}^{(\alpha,F)}\big(A_{n,v}^{(\alpha,F)}\big) \geqslant \mathbf{0},\end{aligned}$$

where equality holds if and only if $X = A_{n,v}^{(\alpha,F)}$. If (i) is satisfied, then (ii) immediately follows. Conversely, assume that (ii) holds. Then Theorem 18 yields

$$\varphi_{v,n}^{(\alpha,F)}\big(A_{n,v}^{(\alpha,F)}\big) = \varphi_{v,n+m}^{(\alpha,F)}\big(A_{n+m,v}^{(\alpha,F)}\big).$$

Consequently, from (43) and Remark 7 we get (i). The implication "(i) $\Rightarrow$ (iii)" is trivial. Conversely, assuming that (iii) is satisfied we obtain (i) from (43) and Theorem 18. $\square$

Theorem 22. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\setminus\mathbb{T}$, let n and m be integers which satisfy $0 \leqslant m \leqslant n$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Further, let $(v_k)_{k=0}^{n}$ be a sequence of pairwise different points which belong to $\mathbb{C}_0\setminus P_{\alpha,n}$, let $(\mathbf{C}_k)_{k=0}^{m}$ be a sequence of complex $q \times p$ matrices, and let*

$$\mathbf{C} := \left(\mathbf{C}_0^*, \mathbf{C}_1^*, \dots, \mathbf{C}_m^*\right)^*.$$

(a) *The matrix-valued function X given by*

$$X := \left(A_{n,v_0}^{(\alpha,F)}, A_{n,v_1}^{(\alpha,F)}, \dots, A_{n,v_m}^{(\alpha,F)}\right)\left[\left(K_{n;r}^{(\alpha,F)}(v_j, v_k)\right)_{j,k=0}^{m}\right]^{-1}\mathbf{C}$$

belongs to $\mathscr{R}_{\alpha,n}^{q\times p}$ and satisfies

$$X(v_k) = \mathbf{C}_k \tag{44}$$

for each $k \in \mathbb{N}_{0,m}$.

(b) *For each $Y \in \mathscr{R}_{\alpha,n}^{q\times p}$ which satisfies*

$$Y(v_k) = \mathbf{C}_k \tag{45}$$

for each $k \in \mathbb{N}_{0,n}$, the inequality

$$\int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{Y} \geqslant \mathbf{C}^*\left[\left(K_{n;r}^{(\alpha,F)}(v_j, v_k)\right)_{j,k=0}^{m}\right]^{-1}\mathbf{C}$$

is true, where equality holds if and only if $Y = X$.

Proof. By virtue of Remark 14 the matrix

$$\mathbf{K} := \left(K_{n;r}^{(\alpha,F)}(v_j, v_k)\right)_{j,k=0}^{m}$$

is positive Hermitian. Since $A_{n,v_k}^{(\alpha,F)} \in \mathscr{R}_{\alpha,n}^{q\times q}$ holds for each $k \in \mathbb{N}_{0,n}$ we see then that X is a well-defined matrix-valued function which belongs to $\mathscr{R}_{\alpha,n}^{q\times p}$. Furthermore, we have

$$\begin{pmatrix} X(v_0) \\ X(v_1) \\ \vdots \\ X(v_m) \end{pmatrix} = \left(A_{n,v_k}^{(\alpha,F)}(v_j)\right)_{j,k=0}^{m} \quad \mathbf{K}^{-1}\mathbf{C} = \mathbf{C},$$

i.e., (44) is satisfied for each $k \in \mathbb{N}_{0,m}$. Using

$$\int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X} = \mathbf{C}^*\mathbf{K}^{-*} \cdot \left(\int_{\mathbb{T}} \left(A_{n,v_j}^{(\alpha,F)}\right)^* \mathrm{d}F A_{n,v_k}^{(\alpha,F)}\right)_{j,k=0}^{m} \cdot \mathbf{K}^{-1}\mathbf{C}$$

and (34) for X e obtain the equality asserted in part (b). Now let us consider an arbitrary $Y \in \mathscr{R}_{\alpha,n}^{q\times p}$ which satisfies (45) for each $k \in \mathbb{N}_{0,m}$. Then $Z := Y - X$ also belongs to $\mathscr{R}_{\alpha,n}^{q\times p}$ and fulfills $Z(v_k) = \mathbf{0}$ for each $k \in \mathbb{N}_{0,m}$. A standard result of the integration theory concerning nonnegative Hermitian measures and Theorem 10 provide us

$$\int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{Z} = \mathbf{C}^* \mathbf{K}^{-*} \begin{pmatrix} \int_{\mathbb{T}} \left(\underline{A_{n,v_0}^{(\alpha,F)}}\right)^* \mathrm{d}F \underline{Z} \\ \int_{\mathbb{T}} \left(\underline{A_{n,v_1}^{(\alpha,F)}}\right)^* \mathrm{d}F \underline{Z} \\ \vdots \\ \int_{\mathbb{T}} \left(\underline{A_{n,v_m}^{(\alpha,F)}}\right)^* \mathrm{d}F \underline{Z} \end{pmatrix} = \mathbf{C}^* \mathbf{K}^{-*} \begin{pmatrix} Z(v_0) \\ Z(v_1) \\ \vdots \\ Z(v_m) \end{pmatrix} = \mathbf{0}$$

and therefore

$$\begin{aligned} \int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{Y} &= \int_{\mathbb{T}} (\underline{X} + \underline{Z})^* \mathrm{d}F (\underline{X} + \underline{Z}) \\ &= \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X} + \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{Z} + \left(\int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{Z} \right)^* + \int_{\mathbb{T}} \underline{Z}^* \mathrm{d}F \underline{Z} \\ &= \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X} + \int_{\mathbb{T}} \underline{Z}^* \mathrm{d}F \underline{Z} \\ &\geqslant \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X}. \end{aligned}$$

Obviously,

$$\int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F \underline{Y} = \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X} \quad \text{if and only if} \quad \int_{\mathbb{T}} \underline{Z}^* \mathrm{d}F \underline{Z} = \mathbf{0}.$$

In view of Remark 7 the proof is complete. □

Corollary 23. *Let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C} \setminus \mathbb{T}$, let $n \in \mathbb{N}_0$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Let $v \in \mathbb{C}_0 \setminus P_{\alpha,n}$ and let $Y \in \mathscr{R}_{\alpha,n}^{q \times p}$ be such that $Y(v) = \mathbf{0}$. Further, let $\mathbf{C} \in \mathbb{C}^{q \times p}$. Then*

$$\int_{\mathbb{T}} (\mathbf{C} - \underline{Y})^* \mathrm{d}F (\mathbf{C} - \underline{Y}) \geqslant \mathbf{C}^* \left(K_{n;r}^{(\alpha,F)}(v, v) \right)^{-1} \mathbf{C}$$

where equality holds if and only if $Y = \mathbf{C} - A_{n,v}^{(\alpha,F)} (K_{n;r}^{(\alpha,F)}(v, v))^{-1} \mathbf{C}$.

Proof. Obviously, $X := \mathbf{C} - Y$ belongs to $\mathscr{R}_{\alpha,n}^{q \times p}$ and satisfies $X(v) = \mathbf{C}$. Application of Theorem 22 yields immediately the assertion. □

Observe that under the assumptions of Corollary 23 in the right $\mathbb{C}^{p \times p}$-Hilbert module $\left(\mathscr{R}_{\alpha,n}^{q \times p}, (\cdot, \cdot)_r\right)$ the function $Z := \mathbf{C} - A_{n,v}^{(\alpha,F)} \left(K_{n;r}^{(\alpha,F)}(v, v) \right)^{-1} \mathbf{C}$ is exactly the orthogonal projection of the constant function with value $\mathbf{C}$ onto the $\mathbb{C}^{p \times p}$-submodule of all $Y \in \mathscr{R}_{\alpha,n}^{q \times p}$ which satisfy $Y(v) = \mathbf{0}$.

Proposition 24. *Let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C} \setminus \mathbb{T}$, let $n \in \mathbb{N}_0$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Further, let $X \in \mathscr{R}_{\alpha,n}^{q \times p}$ be such that*

$$\int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X} = \mathbf{I}. \tag{46}$$

(a) *For each* $v \in \mathbb{C}_0 \backslash P_{\alpha,n}$,

$$K_{n;r}^{(\alpha,F)}(v, v) \geqslant X(v)X^*(v). \tag{47}$$

(b) *Suppose* $p = q$ *and let* $v \in \mathbb{C}_0 \backslash P_{\alpha,n}$. *For each unitary complex* $q \times q$ *matrix* $\mathbf{U}$, *the matrix-valued function*

$$Y_{\mathbf{U}} := A_{n,v}^{(\alpha,F)} \sqrt{K_{n;r}^{(\alpha,F)}(v, v)}^{-1} \mathbf{U} \tag{48}$$

belongs to $\mathscr{R}_{\alpha,n}^{q \times q}$ *and satisfies*

$$\int_{\mathbb{T}} \underline{Y_{\mathbf{U}}}^* \mathrm{d}F \underline{Y_{\mathbf{U}}} = \mathbf{I}. \tag{49}$$

Moreover, equality holds in (47) *if and only if there is a unitary complex* $q \times q$ *matrix* $\mathbf{U}$ *such that* $X = Y_{\mathbf{U}}$.

Proof. Let $v \in \mathbb{C}_0 \backslash P_{\alpha,n}$. In view of $\left(X, A_{n,v}^{(\alpha,F)}\right) \in \mathscr{R}_\alpha^{q \times (p+q)}$ we have

$$\begin{pmatrix} \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{X} & \int_{\mathbb{T}} \underline{X}^* \mathrm{d}F \underline{A_{n,v}^{(\alpha,F)}} \\ \int_{\mathbb{T}} \left(\underline{A_{n,v}^{(\alpha,F)}}\right)^* \mathrm{d}F \underline{X} & \int_{\mathbb{T}} \left(\underline{A_{n,v}^{(\alpha,F)}}\right)^* \mathrm{d}F \underline{A_{n,v}^{(\alpha,F)}} \end{pmatrix}$$
$$= \int_{\mathbb{T}} \underline{\left(X, A_{n,v}^{(\alpha,F)}\right)}^* \mathrm{d}F \underline{\left(X, A_{n,v}^{(\alpha,F)}\right)} \geqslant \mathbf{0}. \tag{50}$$

From Theorem 10 we know that

$$\int_{\mathbb{T}} \left(\underline{A_{n,v}^{(\alpha,F)}}\right)^* \mathrm{d}F \underline{X} = X(v) \tag{51}$$

and

$$\int_{\mathbb{T}} \left(\underline{A_{n,v}^{(\alpha,F)}}\right)^* \mathrm{d}F \underline{A_{n,v}^{(\alpha,F)}} = A_{n,v}^{(\alpha,F)}(v) = K_{n;r}^{(\alpha,F)}(v, v) \tag{52}$$

are true. Hence from (50), (46), (51) and (52) we see that the matrix

$$\begin{pmatrix} \mathbf{I} & X^*(v) \\ X(v) & K_{n;r}^{(\alpha,F)}(v, v) \end{pmatrix}$$

is nonnegative Hermitian. Using a well-known characterization of nonnegative Hermitian block matrices given in [1,8] (see also [7, Lemma 1.1.9] inequality (47) follows. It remains to check part (b). Assume $p = q$. If $\mathbf{U}$ is a unitary complex $q \times q$ matrix, then the matrix-valued function $Y_{\mathbf{U}}$ defined by (48) belongs obviously to $\mathscr{R}_{\alpha,n}^{q \times q}$ and satisfies

$$Y_{\mathbf{U}}(v) = \sqrt{K_{n;r}^{(\alpha,F)}(v, v)} \mathbf{U}. \tag{53}$$

Standard arguments of the integration theory with respect to nonnegative Hermitian measures and Theorem 10 yield

$$\begin{aligned}\int_{\mathbb{T}} \underline{Y_{\mathbf{U}}}^* \mathrm{d}F \underline{Z} &= \mathbf{U}^* \sqrt{K_{n;r}^{(\alpha,F)}(v,v)}^{-1} \int_{\mathbb{T}} \underline{\left(A_{n,v}^{(\alpha,F)}\right)^*} \mathrm{d}F \underline{Z} \\ &= \mathbf{U}^* \sqrt{K_{n;r}^{(\alpha,F)}(v,v)}^{-1} Z(v) \end{aligned} \tag{54}$$

for each $Z \in \mathscr{R}_{\alpha,n}^{q\times q}$. In particular, (54) and (53) give then

$$\int_{\mathbb{T}} \underline{Y_{\mathbf{U}}}^* \mathrm{d}F \underline{Y_{\mathbf{U}}} = \mathbf{U}^* \sqrt{K_{n;r}^{(\alpha,F)}(v,v)}^{-1} Y_{\mathbf{U}}(v) = \mathbf{U}^*\mathbf{U} = \mathbf{I}, \tag{55}$$

i.e., (49) is satisfied. Furthermore, (53) implies immediately $Y_{\mathbf{U}}(v)Y_{\mathbf{U}}^*(v) = K_{n;r}^{(\alpha,F)}(v,v)$. Conversely, now assume that X fulfills $X(v)X^*(v) = K_{n;r}^{(\alpha,F)}(v,v)$. The Polar Decomposition Theorem shows then that there is a unitary complex $q \times q$ matrix $\mathbf{U}$ such that

$$X(v) = \sqrt{K_{n;r}^{(\alpha,F)}(v,v)}\,\mathbf{U}.$$

Using (54) we get then

$$\int_{\mathbb{T}} \underline{Y_{\mathbf{U}}}^* \mathrm{d}F \underline{X} = \mathbf{U}^*\mathbf{U} = \mathbf{I}.$$

Thus from (46), (49) and (55) we can conclude

$$\begin{aligned}&\int_{\mathbb{T}} (\underline{X} - \underline{Y_{\mathbf{U}}})^* \mathrm{d}F (\underline{X} - \underline{Y_{\mathbf{U}}}) \\ &= \int_{\mathbb{T}} (\underline{X}^* \mathrm{d}F \underline{X}) - \int_{\mathbb{T}} \underline{Y_{\mathbf{U}}}^* \mathrm{d}F \underline{X} - \left(\int_{\mathbb{T}} \underline{Y_{\mathbf{U}}}^* \mathrm{d}F \underline{X}\right)^* + \int_{\mathbb{T}} \underline{Y_{\mathbf{U}}}^* \mathrm{d}F \underline{Y_{\mathbf{U}}} = \mathbf{0}.\end{aligned}$$

From Remark 7 it follows $X = Y_{\mathbf{U}}$. □

5. A particular solution of the rational moment problem

In this section, under some nondegeneracy condition for the given data, we will give a solution of the rational moment problem (R). If $w \in \mathbb{C}$, then let $f_w : \mathbb{C} \to \mathbb{C}$ and $g_w : \mathbb{C} \to \mathbb{C}$ be defined by $f_w(z) := z - w$ and $g_w(z) := 1 - \overline{w}z$.

Theorem 25. *Let $n \in \mathbb{N}_0$ and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. If $n > 0$, then let $(\alpha_j)_{j=1}^n$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$. For every choice of z in $\mathbb{T}$ and v in $\mathbb{C}\backslash(\mathbb{T} \cup P_{\alpha,n})$, the matrices $K_{n;r}^{(\alpha,F)}(z,v)$ and $K_{n;r}^{(\alpha,F)}(v,z)$ are both nonsingular.*

Proof. We modify an idea which was used by Delsarte et al. in [4, Theorem 6]. The case $n = 0$ is trivial. Suppose $n > 0$. Let $v \in \mathbb{C}\backslash(\mathbb{T} \cup P_{\alpha,n})$. From Corollary 19 we know that the matrix $K_{n;r}^{(\alpha,F)}(v,v)$ is positive Hermitian. Now we assume that $w_0 \in \mathbb{C}\setminus(\{0\} \cup P_{\alpha,n})$ is such that the matrix $K_{n;r}^{(\alpha,F)}(w_0, v)$ is singular. Then $w_0 \neq v$ and there is a nontrivial column vector $\mathbf{u} = (u_1, u_2, \ldots, u_q)^{\mathrm{T}} \in \mathbb{C}^q$ such that $K_{n;r}^{(\alpha,F)}(w_0, v)\mathbf{u} = \mathbf{0}$. For each integer j satisfying $1 \leqslant j \leqslant n-1$ let $\gamma_j := \alpha_{j+1}$. Since $A_{n,v}^{(\alpha,F)}$ belongs to $\mathscr{R}_{\alpha,n}^{q\times q}$ there is a $U \in \mathscr{R}_{\gamma,n-1}^{q\times 1}$ such that

$$A_{n,v}^{(\alpha,F)}\mathbf{u} = \frac{1}{w_0}\frac{f_{w_0}}{g_{\alpha_1}}U = \frac{f_{w_0}}{w_0}Y \tag{56}$$

where $Y := \frac{1}{g_{\alpha_1}}U$ belongs to $\mathscr{R}_{\alpha,n}^{q\times 1}$. In particular,

$$K_{n;r}^{(\alpha,F)}(v, v)\mathbf{u} = A_{n,v}^{(\alpha,F)}(v)\mathbf{u} = \frac{v - w_0}{w_0}Y(v).$$

Since $K_{n;r}^{(\alpha,F)}(v, v)$ is nonsingular and $u \neq \mathbf{0}$, we have $Y(v) \neq \mathbf{0}$. Because of $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ from Remark 7 we obtain then that $\int_{\mathbb{T}} \underline{Y}^* \mathrm{d}F\underline{Y} \in (0, +\infty)$.

Now we consider the function $f : \mathbb{C}\setminus(\{0\} \cup P_{\alpha,n}) \to \mathbb{R}$ given by

$$f(w) := \int_{\mathbb{T}} \left(\frac{1}{w}\underline{f_w Y}\right)^* \mathrm{d}F \left(\frac{1}{w}\underline{f_w Y}\right) - 2\,\mathrm{Re}\left(\frac{w_0(v - w)}{w(v - w_0)}\right)\mathbf{u}^* K_{n;r}^{(\alpha,F)}(v, v)\mathbf{u}.$$

We want to express f in such a form that we see the global minimum of f. Let $w \in \mathbb{C}\setminus(\{0\} \cup P_{\alpha,n})$. Since the column vector $\mathbf{u}$ is nontrivial there is a $k \in \mathbb{N}_{1,q}$ such that $u_k \neq 0$. Let $\mathbf{a}_k^{(q)} := (\delta_{1,k}, \delta_{2,k}, \ldots, \delta_{q,k})$ where δ_{jk} is the Kronecker symbol. Then

$$X_0 := \frac{f_w}{w u_k} Y \mathbf{a}_k^{(q)}$$

belongs to $\mathscr{R}_{\alpha,n}^{q\times q}$ and satisfies

$$X_0\mathbf{u} = \frac{f_w}{w}Y.$$

If X is an arbitrary matrix-valued function which belongs to $\mathscr{R}_{\alpha,n}^{q\times q}$ and which satisfies $X\mathbf{u} = \frac{f_w}{w}Y$, from (56) one can easily see then that

$$X(v)\mathbf{u} = \frac{w_0(v - w)}{w(v - w_0)} A_{n,v}^{(\alpha,F)}(v)\mathbf{u}$$

which implies

$$2\mathbf{u}^* \,\mathrm{Re}[X(v)]\mathbf{u} = \left[2\,\mathrm{Re}\left(\frac{w_0(v - w)}{w(v - w_0)}\right)\right]\mathbf{u}^* K_{n;r}^{(\alpha,F)}(v, v)\mathbf{u}$$

and consequently $f(w) = \mathbf{u}^*\varphi_{v,n}^{(\alpha,F)}(X)\mathbf{u}$. According to (56) we have in particular $f(w_0) = \mathbf{u}^*\varphi_{v,n}^{(\alpha,F)}\big(A_{n,v}^{(\alpha,F)}\big)\mathbf{u}$. Applying Theorem 18 we get then

$$f(w) \geqslant f(w_0) \tag{57}$$

and $f(w_0) = -\mathbf{u}^* K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}$. This equation immediately implies

$$\int_{\mathbb{T}} \left(\frac{1}{w_0} \underline{f_{w_0} Y}\right)^* \mathrm{d}F \left(\frac{1}{w_0} \underline{f_{w_0} Y}\right) = \mathbf{u}^* K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}. \tag{58}$$

Let $\mathscr{S}$ denote the set of all $\binom{x}{y} \in \mathbb{R}^2$ for which $x + \mathrm{i}y$ belongs to $\{0\} \cup P_{\alpha,n}$, and let $g : \mathbb{R}^2 \backslash \mathscr{S} \to \mathbb{R}$ be defined by

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) := f(x + \mathrm{i}y).$$

Then

$$\begin{aligned} g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = {} & \left(1 + \frac{1}{x^2 + y^2}\right) \int_{\mathbb{T}} \underline{Y}^*(z) F(\mathrm{d}z) \underline{Y}(z) \\ & - \frac{1}{x + \mathrm{i}y} \int_{\mathbb{T}} \underline{Y}^*(z) F(\mathrm{d}z)(z\underline{Y}(z)) - \frac{1}{x - \mathrm{i}y} \int_{\mathbb{T}} (z\underline{Y}(z))^* \mathrm{d}F \underline{Y}(z) \\ & - \left(\frac{w_0}{w_0 - v}\left(1 - \frac{v}{x + \mathrm{i}y}\right) + \overline{\left(\frac{w_0}{w_0 - v}\right)}\left(1 - \frac{\overline{v}}{x - \mathrm{i}y}\right)\right) \\ & \times \mathbf{u}^* K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u} \end{aligned}$$

holds for all $\binom{x}{y} \in \mathbb{R}^2 \backslash \mathscr{S}$. Hence, for each $\binom{x}{y} \in \mathbb{R}^2 \backslash \mathscr{S}$, we have

$$\begin{aligned} & \frac{\partial g}{\partial x}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ & \quad = \frac{-2x}{(x^2 + y^2)^2} \int_{\mathbb{T}} \underline{Y}^*(z) F(\mathrm{d}z) \underline{Y}(z) + \frac{1}{(x + \mathrm{i}y)^2} \int_{\mathbb{T}} \underline{Y}^*(z) F(\mathrm{d}z)(z\underline{Y}(z)) \\ & \quad + \frac{1}{(x - \mathrm{i}y)^2} \int_{\mathbb{T}} (z\underline{Y}(z))^* F(\mathrm{d}z) \underline{Y}(z) \\ & \quad - \left(\frac{w_0 v}{(w_0 - v)(x + \mathrm{i}y)^2} + \frac{\overline{w_0 v}}{(\overline{w_0} - \overline{v})(x - \mathrm{i}y)^2}\right) \mathbf{u}^* K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u} \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial g}{\partial y}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ & \quad = \frac{-2y}{(x^2 + y^2)^2} \int_{\mathbb{T}} \underline{Y}^*(z) F(\mathrm{d}z) \underline{Y}(z) + \frac{\mathrm{i}}{(x + \mathrm{i}y)^2} \int_{\mathbb{T}} \underline{Y}^*(z) F(\mathrm{d}z)(z\underline{Y}(z)) \\ & \quad - \frac{\mathrm{i}}{(x - \mathrm{i}y)^2} \int_{\mathbb{T}} (z\underline{Y}(z))^* F(\mathrm{d}z) \underline{Y}(z) \\ & \quad - \left(\frac{\mathrm{i} w_0 v}{(w_0 - v)(x + \mathrm{i}y)^2} - \frac{\mathrm{i}\overline{w_0 v}}{(\overline{w_0} - \overline{v})(x - \mathrm{i}y)^2}\right) \mathbf{u}^* K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}. \end{aligned}$$

Because of (57) we have

$$
\begin{aligned}
0 &= \frac{\partial g}{\partial x}\left(\begin{pmatrix}\operatorname{Re} w_0\\ \operatorname{Im} w_0\end{pmatrix}\right) + \mathrm{i}\frac{\partial g}{\partial y}\left(\begin{pmatrix}\operatorname{Re} w_0\\ \operatorname{Im} w_0\end{pmatrix}\right)\\
&= -\frac{2w_0}{|w_0|^4}\int_{\mathbb{T}} \underline{Y}^*(z)F(\mathrm{d}z)\underline{Y}(z) + \frac{2}{\overline{w_0}^2}\int_{\mathbb{T}}(z\underline{Y}(z))^*F(\mathrm{d}z)\underline{Y}(z)\\
&\quad - \frac{2\overline{v}}{(\overline{w_0}-\overline{v})\overline{w_0}}\mathbf{u}^*K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}\\
&= -\frac{2w_0}{|w_0|^4}\left(\int_{\mathbb{T}}(1-\overline{z}w_0)\underline{Y}^*(z)F(\mathrm{d}z)\underline{Y}(z) + \frac{|w_0|^2\overline{v}}{\overline{w_0}-\overline{v}}\mathbf{u}^*K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}\right).
\end{aligned}
\tag{59}
$$

Obviously,

$$
2\operatorname{Re}(1-\overline{z}w_0) = 1-|w_0|^2 + (\overline{z}+\overline{w_0})(z-w_0) = 1-|w_0|^2 + \overline{f_{w_0}(z)}f_{w_0}(z)
$$

holds for each $z \in \mathbb{T}$. Further, the identity

$$
2\operatorname{Re}\frac{|w_0|^2\overline{v}}{\overline{w_0}+\overline{v}} = \frac{|w_0|^2}{|w_0-v|^2}\left(\overline{v}w_0 + v\overline{w_0} - 2|v|^2\right)
$$

is true. Thus using (59) and elementary results of the integration theory with respect to nonnegative Hermitian measures we can conclude

$$
\begin{aligned}
0 &= 2\operatorname{Re}\left(\int_{\mathbb{T}}(1-\overline{z}w_0)\underline{Y}^*(z)F(\mathrm{d}z)\underline{Y}(z)\right) + 2\operatorname{Re}\left(\frac{|w_0|^2\overline{v}}{\overline{w_0}-\overline{v}}\mathbf{u}^*K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}\right)\\
&= \int_{\mathbb{T}}(2\operatorname{Re}(1-\overline{z}w_0))\,\underline{Y}^*(z)F(\mathrm{d}z)\underline{Y}(z) + \left(2\operatorname{Re}\frac{|w_0|^2\overline{v}}{\overline{w_0}-\overline{v}}\right)\mathbf{u}^*K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}\\
&= (1-|w_0|^2)\int_{\mathbb{T}}\underline{Y}^*\mathrm{d}F\underline{Y} + |w_0|^2\int_{\mathbb{T}}\left[\frac{1}{\underline{w_0}}f_{w_0}(z)\underline{Y}(z)\right]^*F(\mathrm{d}z)\left[\frac{1}{\underline{w_0}}f_{w_0}(z)\underline{Y}(z)\right]\\
&\quad - \frac{|w_0|^2\left(2|v|^2-\overline{v}w_0-v\overline{w_0}\right)}{|w_0-v|^2}\mathbf{u}^*K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}.
\end{aligned}
\tag{60}
$$

Using (58) we get then

$$
0 = (1-|w_0|^2)\int_{\mathbb{T}}\underline{Y}^*\mathrm{d}F\underline{Y} + \frac{|w_0|^2(|w_0|^2-|v|^2)}{|w_0-v|^2}\mathbf{u}^*K_{n;r}^{(\alpha,F)}(v,v)\mathbf{u}
$$

where, in view of Corollary 19, the matrix $K_{n;r}^{(\alpha,F)}(v,v)$ is positive Hermitian. Hence, if w_0 would belong to the unit circle $\mathbb{T}$, it would follow that v belongs to $\mathbb{T}$ as well, which contradicts the assumption. Consequently, $w_0 \notin \mathbb{T}$. Finally, we observe that, for every choice of z in $\mathbb{T}$, the equation $K_{n;r}^{(\alpha,F)}(z,v) = \left[K_{n;r}^{(\alpha,F)}(v,z)\right]^*$ is satisfied for each $v \in \mathbb{C}\backslash P_{\alpha,n}$. $\square$

In the matrix polynomial case it has proved to be useful to consider so-called rational approximation measures associated with nonnegative Hermitian Borel measures on the unit circle $\mathbb{T}$ (see [4] and [7, Section 3.6]). We want to give an analogue for the rational situation studied here. The notation $\underline{\lambda}$ stands for the linear Lebesgue–Borel measure on $\mathbb{T}$.

Proposition 26. *Let $w \in \mathbb{D}$, let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, and let $F \in \mathscr{M}_{\geqslant}^{q;n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.*

(a) *The matrix-valued function $V_{n,w}^{(\alpha,F)} : \mathbb{T} \to \mathbb{C}^{q\times q}$ given by*

$$V_{n,w}^{(\alpha,F)}(z) := \frac{1-|w|^2}{|z-w|^2}\big[K_{n;r}^{(\alpha,F)}(z,w)\big]^{-*}K_{n;r}^{(\alpha,F)}(w,w)\big[K_{n;r}^{(\alpha,F)}(z,w)\big]^{-1}$$

is continuous. For each $z \in \mathbb{T}$, the matrix $V_{n,w}^{(\alpha,F)}(z)$ is positive Hermitian.

(b) *The mapping $F_{n,w}^{[\alpha]} : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q\times q}$ defined by*

$$F_{n,w}^{[\alpha]}(B) := \frac{1}{2\pi}\int_B V_{n,w}^{(\alpha,F)}\,\mathrm{d}\underline{\lambda} \tag{61}$$

belongs to $\mathscr{M}_{\geqslant}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

Proof. Part (a) is an immediate consequence of Theorem 25 and Corollary 19. Furthermore, the assertion stated in part (b) follows from part (a) and [9, Example 5.4]. □

Observe that, under the assumptions of Proposition 26, the matrix-valued function $V_{n,w}^{(\alpha,F)}$ can be represented via

$$V_{n,w}^{(\alpha,F)}(z) = \frac{1-|w|^2}{|z-w|^2}\big[K_{n;l}^{(\alpha,F)}(w,z)\big]^{-1}K_{n;l}^{(\alpha,F)}(w,w)[K_{n;l}(w,z)]^{-*}$$

for all $z \in \mathbb{T}$. However, this identity will not be used in this paper so that we omit a proof. In the special case $w = 0$, we have

$$V_{n,0}^{(\alpha,F)}(z) = \big[A_{n,0}^{(\alpha,F)}(z)\big]^{-*}A_{n,0}^{(\alpha,F)}(0)\big[A_{n,0}^{(\alpha,F)}(z)\big]^{-1}$$

for each $z \in \mathbb{T}$. Thus, in view of Remark 9, one can easily see that in the particular case that $\alpha_j = 0$ holds for each integer j satisfying $1 \leqslant j \leqslant n$ the nonnegative Hermitian measure $F_{n,0}^{[\alpha]}$ is exactly the nth rational approximation measure considered in the polynomial case (see [4] and [7, Section 3.6]).

For each positive real number η, let $\mathscr{K}(0;\eta) := \{z \in \mathbb{C} : |z| < \eta\}$.

Lemma 27. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, let $w \in \mathbb{D}$, and let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Further, let F_a denote the absolutely continuous part of F in the Lebesgue decomposition with respect to $\frac{1}{2\pi}\underline{\lambda}$. Moreover, suppose that there are a real number η with $\eta > 1$ and a complex $q \times q$ matrix-valued function G defined on $\mathscr{K}(0;\eta)$ such that the following three conditions are satisfied*:

(i) *G is holomorphic in $\mathscr{K}(0;\eta)$.*
(ii) *The matrix $G(w)$ is nonsingular.*
(iii) *For each $B \in \mathfrak{B}_{\mathbb{T}}$,*

$$F_a(B) = \frac{1}{2\pi}\int_B \frac{1-|w^2|}{|z-w|^2}\underline{G}^*(z)\underline{G}(z)\underline{\lambda}(\mathrm{d}z).$$

For each $X \in \mathscr{R}^{q\times q}_{\alpha,n}$, then

$$\varphi^{(\alpha,F)}_{w,n}(X) \geqslant \varphi^{(\alpha,F_a)}_{w,n}(X) \geqslant -\mathbf{R}^*\mathbf{R} \tag{62}$$

where $\mathbf{R} := (G(w))^{-*}$.

Proof. Let $X \in \mathscr{R}^{q\times q}_{\alpha,n}$. Then there is an $\eta_0 \in (1,+\infty)$ such that the matrix-valued function $\mathrm{Rstr.}_{\mathscr{K}(0;\eta_0)}GX$ is holomorphic in $\mathscr{K}(0;\eta_0)$. Using the Poisson integral representation we obtain

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|w|^2}{|z-w|^2}\mathbf{R}^*\underline{GX}(z)\underline{\lambda}(\mathrm{d}z) = \mathbf{R}^*(GX)(w) = X(w)$$

and, in view of (iii) and (35), therefore

$$\begin{aligned}\varphi^{(\alpha,F_a)}_{w,n}(X) + \mathbf{R}^*\mathbf{R} &= \int_{\mathbb{T}} \underline{X}^*\mathrm{d}F_a\underline{X} - X(w) - [X(w)]^* + \mathbf{R}^*\mathbf{R}\\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|w|^2}{|z-w|^2}[\mathbf{R}-\underline{GX}(z)]^*[\mathbf{R}-\underline{GX}(z)]\underline{\lambda}(\mathrm{d}z) \geqslant \mathbf{0}.\end{aligned}$$

Elementary statements of the integration theory with respect to nonnegative Hermitian measures provide the first inequality in (62). □

Remark 28. Let $F \in \mathscr{M}^q_{\geqslant}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$, let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers belonging to $\mathbb{C}\backslash\mathbb{T}$, and let $n \in \mathbb{N}_0$. Then the nonnegative Hermitian Borel measure $F^{(\alpha,n)}$ on $\mathbb{T}$ given by (20) satisfies $\mathbf{T}_n^{(F^{(\alpha,n)})} = \tilde{\mathbf{G}}_n^{(\alpha,F)}$. Moreover, $F \in \mathscr{M}^{q,n}_{\geqslant}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ if and only if $F^{(\alpha,n)} \in \mathscr{M}^{q,n}_{\geqslant}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ (see [9, Remark 5.10]).

Remark 29. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$, let $n \in \mathbb{N}_0$, and let $F \in \mathscr{M}^{q,n}_{\geqslant}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$. In view of Remark 28, one can easily see then that, for every choice of v and w in $\mathbb{C}_0\backslash P_{\alpha,n}$,

$$K^{(\alpha,F)}_{n;r}(v,w) = \left(\left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](v)\right)\left(\mathbf{T}_n^{(F^{(\alpha,n)})}\right)^{-1}\left(\left[\frac{1}{\pi_{\alpha,n}}e_n^{(q)}\right](w)\right)^*.$$

Moreover,

$$A^{(\alpha,F)}_{n,0} = \frac{1}{\pi_{\alpha,n}}A_n^{[F^{(\alpha,n)}]} \tag{63}$$

where the matrix polynomial $A_n^{[F^{(\alpha,n)}]}$ is given by (29).

Now we will continue to use the notations introduced in Proposition 26.

Theorem 30. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of numbers belonging to $\mathbb{C}\setminus\mathbb{T}$, let $n \in \mathbb{N}_0$, and let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Then*

$$\tilde{\mathbf{G}}_j^{\left(\alpha, F_{n,0}^{[\alpha]}\right)} = \tilde{\mathbf{G}}_j^{(\alpha,F)}$$

for all $j \in \mathbb{N}_{0,n}$. In particular,

$$A_{j,v}^{\left(\alpha, F_{n,0}^{[\alpha]}\right)} = A_{j,v}^{(\alpha,F)} \tag{64}$$

for each $j \in \mathbb{N}_{0,n}$ and each $v \in \mathbb{C}_0\setminus P_{\alpha,j}$. Moreover,

$$A_{k,0}^{\left(\alpha, F_{n,0}^{[\alpha]}\right)} = A_{n,0}^{(\alpha,F)} \tag{65}$$

for each integer k with $k \geqslant n$.

Proof. First observe that from Remark 28 we know that $F^{(\alpha,n)}$ belongs to $\mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. From Remark 29 we then see that the identity (63) is satisfied. Therefore application of Theorem 25 shows that, for each $z \in \mathbb{T}$, the matrix $A_{n,0}^{(\alpha,F)}(z)$ is nonsingular. Hence we get

$$\begin{aligned} V_{n,0}^{(\alpha,F)}(z) &= \left(A_{n,0}^{(\alpha,F)}(z)\right)^{-*} A_{n,0}^{(\alpha,F)}(0)\left(A_{n,0}^{(\alpha,F)}(z)\right)^{-1} \\ &= (\pi_{\alpha,n}(z)\mathbf{I}_q)^*\left(A_n^{[F^{(\alpha,n)}]}(z)\right)^{-*} A_n^{[F^{(\alpha,n)}]}(0)\left(A_n^{[F^{(\alpha,n)}]}(z)\right)^{-1}(\pi_{\alpha,n}(z)\mathbf{I}_q) \end{aligned}$$

for each $z \in \mathbb{T}$. According to Proposition 26 the measure $F_{n,0}^{[\alpha]}$ introduced in (61) belongs to $\mathscr{M}_{\geqslant}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Using the notation given in (20) and elementary results of the integration theory with respect to nonnegative Hermitian measures we infer

$$\begin{aligned} \left(F_{n,0}^{[\alpha]}\right)^{(\alpha,n)}(B) &= \frac{1}{2\pi}\int_B \left(\underline{\frac{1}{\pi_{\alpha,n}}\mathbf{I}_q}\right)^* V_{n,0}^{(\alpha,F)} \left(\underline{\frac{1}{\pi_{\alpha,n}}\mathbf{I}_q}\right) \mathrm{d}\underline{\lambda} \\ &= \frac{1}{2\pi}\int_B \left(\underline{A_n^{[F^{(\alpha,n)}]}}(z)\right)^{-*} A_n^{[F^{(\alpha,n)}]}(0)\left(\underline{A_n^{[F^{(\alpha,n)}]}}(z)\right)^{-1}\underline{\lambda}(\mathrm{d}z) \end{aligned}$$

for each $B \in \mathfrak{B}_{\mathbb{T}}$, i.e., $(F_{n,0}^{[\alpha]})^{(\alpha,n)}$ is exactly the nth rational approximation measure associated with $F^{(\alpha,n)}$ which is considered in the matrix polynomial case (see [7, Section 3.6]). Using Remark 28 and a result from the Szegö theory of orthogonal matrix polynomials (see, e.g., [7, Proposition 3.6.7]) we obtain

$$\tilde{\mathbf{G}}_n^{\left(\alpha, F_{n,0}^{[\alpha]}\right)} = \mathbf{T}_n^{\left(\left(F_{n,0}^{[\alpha]}\right)^{(\alpha,n)}\right)} = \mathbf{T}_n^{(F^{(\alpha,n)})} = \tilde{\mathbf{G}}_n^{(\alpha,F)}. \tag{66}$$

Now we consider an arbitrary nonnegative integer j with $j \leqslant n$. Clearly, $\mathscr{R}_{\alpha,j}^{q\times q} \subseteq \mathscr{R}_{\alpha,n}^{q\times q}$. Therefore Remark 1 shows, there is a complex $(n+1)q \times (j+1)q$ matrix $\mathbf{C}$ such that $\frac{1}{\pi_{\alpha,j}} e_j^{(q)} = \frac{1}{\pi_{\alpha,n}} e_n^{(q)}\mathbf{C}$. Thus it is readily checked that

$$\tilde{\mathbf{G}}_j^{(\alpha,F)} = \mathbf{C}^*\tilde{\mathbf{G}}_n^{(\alpha,F)}\mathbf{C} \quad \text{and} \quad \tilde{\mathbf{G}}_j^{\left(\alpha,F_{n,0}^{[\alpha]}\right)} = \mathbf{C}^*\tilde{\mathbf{G}}_n^{\left(\alpha,F_{n,0}^{[\alpha]}\right)}\mathbf{C}.$$

From (66) then $\tilde{\mathbf{G}}_j^{(\alpha,F)} = \tilde{\mathbf{G}}_j^{\left(\alpha,F_{n,0}^{[\alpha]}\right)}$ follows. Since $\mathscr{M}_{\geqslant}^{q,n}(\mathbb{T},\mathfrak{B}_{\mathbb{T}}) \subseteq \mathscr{M}_{\geqslant}^{q,j}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ holds (see, e.g., [9, Remark 5.3]), we get from (26) and (28) in particular that (64) holds for each $v \in \mathbb{C}_0\backslash P_{\alpha,j}$. It remains to prove (65). From [4, Theorem 3 and 5] (see also [7, Corollary 3.6.1 and Proposition 3.6.3]) we see that there is a real number $\eta > 1$ such that the rational matrix-valued function

$$f := \sqrt{A_n^{[F^{(\alpha,n)}]}(0)}\,\left(\mathrm{Rstr.}_{\mathscr{K}(0;\eta)}\pi_{\alpha,n}A_n^{[F^{(\alpha,n)}]}\right)^{-1}$$

is holomorphic in $\mathscr{K}(0;\eta)$. Obviously, $f(0) = \sqrt{A_n^{[F^{(\alpha,n)}]}(0)}^{\,-1}$. In particular, the matrix $f(0)$ is positive Hermitian. Clearly, for each $B \in \mathfrak{B}_{\mathbb{T}}$, we have

$$F_{n,0}^{[\alpha]}(B) = \frac{1}{2\pi}\int_B \underline{f}^*\,\underline{f}\,\mathrm{d}\underline{\lambda}.$$

Let $k \in \mathbb{N}_0$ satisfy $k \geqslant n$. For each $X \in \mathscr{R}_{\alpha,k}^{q\times q}$, from Lemma 27 with $w = 0$ we then obtain

$$\varphi_{0,k}^{\left(\alpha,F_{n,0}^{[\alpha]}\right)}(X) \geqslant -(f(0))^{-1}(f(0))^{-*} = -A_n^{[F^{(\alpha,n)}]}(0). \tag{67}$$

In view of Theorem 18, (28), (64) with $j = n$ and (63), the right-hand side of (67) coincides with $\varphi_{0,n}^{\left(\alpha,F_{n,0}^{[\alpha]}\right)}\left(A_{n,0}^{(\alpha,F)}\right)$. Hence in view of Remark 15, we have

$$\varphi_{0,k}^{\left(\alpha,F_{n,0}^{[\alpha]}\right)}(X) \geqslant \varphi_{0,k}^{\left(\alpha,F_{n,0}^{[\alpha]}\right)}\left(A_{n,0}^{(\alpha,F)}\right)$$

for each $X \in \mathscr{R}_{\alpha,k}^{q\times q}$. Because of $\mathscr{R}_{\alpha,n}^{q\times q} \subseteq \mathscr{R}_{\alpha,k}^{q\times q}$, the matrix-valued function $A_{n,0}^{(\alpha,F)}$ belongs to $\mathscr{R}_{\alpha,k}^{q\times q}$. Since $F_{n,0}^{[\alpha]}$ belongs to $\mathscr{M}_{\geqslant}^{q,\infty}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ from Theorem 18 we can finally conclude that $A_{n,0}^{(\alpha,F)} = A_{k,0}^{\left(\alpha,F_{n,0}^{[\alpha]}\right)}$. $\square$

Now we again turn our attention to Problem (R) given in Section 2. There we stated a necessary and sufficient condition for the solvability of this problem. In the case that Problem (R) has a solution and that the given matrix is nonsingular we can now construct explicitly a particular solution.

Theorem 31. *Let $n \in \mathbb{N}$, let $(\alpha_j)_{j=1}^n$ be a sequence of numbers which belong to $\mathbb{C}\backslash\mathbb{T}$, let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathscr{R}_{\alpha,n}^{q\times q}$, and let $\mathbf{G}$ be a nonsingular complex $(n+1)q \times (n+1)q$ matrix such that $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}, (X_k)_{k=0}^n\right]$ is nonempty. Let the matrix-valued function $a_n^{(\alpha)} : \mathbb{C}_0\backslash P_{\alpha,n} \to \mathbb{C}^{q\times q}$ be given by*

$$a_n^{(\alpha)}(z) := X^{[n]}(z)\mathbf{G}^{-1}(X^{[n]}(0))^*.$$

Then $M : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q\times q}$ *defined by*

$$M(B) := \frac{1}{2\pi}\int_B \left(\underline{a_n^{(\alpha)}}(z)\right)^{-*} a_n^{(\alpha)}(0)\left(\underline{a_n^{(\alpha)}}(z)\right)^{-1}\underline{\lambda}(\mathrm{d}z)$$

belongs to $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}, (X_k)_{k=0}^n\right]$.

Proof. By assumption, there is a $q \times q$ nonnegative Hermitian Borel measure F on $\mathbb{T}$ which belongs to $\mathscr{M}\left[(\alpha_j)_{j=1}^n, \mathbf{G}, (X_k)_{k=0}^n\right]$, i.e., F belongs to $\mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and satisfies $\mathbf{G}_{X,n}^{(F)} = \mathbf{G}$. According to Theorem 6, F necessarily belongs to $\mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. In view of $\mathbf{G}_{X,n}^{(F)} = \mathbf{G}$ and Remark 12, it admits the representation $a_n^{(\alpha)} = A_{n,0}^{(\alpha,F)}$. From Theorem 25 we see then that, for each $z \in \mathbb{T}$, the matrix $a_n^{(\alpha)}(z)$ is nonsingular. If the measure $F_{n,0}^{[\alpha]}$ is given by (61), we obtain that M coincides with $F_{n,0}^{[\alpha]}$. Hence Theorem 30 shows that $\tilde{\mathbf{G}}_n^{(\alpha,M)} = \tilde{\mathbf{G}}_n^{(\alpha,F)}$. From Remark 4 then $\mathbf{G}_{X,n}^{(M)} = \mathbf{G}_{X,n}^{(F)} = \mathbf{G}$ finally follows. □

In our considerations we only used the measures introduced in Proposition 26 for the special case $w = 0$. In a forthcoming paper, we want to show that the theory of orthogonal rational matrix-valued functions can be applied in order to verify that this restriction to the case $w = 0$ can be omitted.

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